chapter 3

Applications of Differential Calculus

Section 3.6 Related rates of change

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In many situations, we are given one rate of change and asked to find another rate of change. In these situations, the *Chain Rule* is particularly useful when finding the required rate of change.

The Chain Rule states that $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$.

This can be extended to $\dfrac{dy}{dx} = \dfrac{dy}{d\bullet} \cdot \dfrac{d\bullet}{dx}$, where $\bullet$ stands for any variable.

For example, $\dfrac{dA}{dr} = \dfrac{dA}{\mathbf{d}\boldsymbol{t}} \cdot \dfrac{\mathbf{d}\boldsymbol{t}}{dr}$

If we are given that $\dfrac{dA}{dr} = 8$, then $\dfrac{dr}{dA} = \dfrac{1}{8}$.

Similarly, if $\dfrac{dV}{dt} = 6\pi$, then $\dfrac{dt}{dV} = \dfrac{1}{6\pi}$.

$$\frac{dt}{dr} = \frac{1}{\frac{dr}{dt}}$$

The method of solving problems involving 'related rates' is illustrated in the following examples.

Example 1

The radius of a circle is increasing at the rate of 2 cm/sec.
Find the rate at which the area is increasing when the radius is 3 cm.

Area of Circle

$A = \pi R^2$

Rate of change

"$\frac{d\bullet}{dt}$"

$R = 3$

$\frac{dR}{dt} = 2$

$\left.\frac{dA}{dt}\right|_{r=3} = ?$

$\frac{dA}{dR} = 2\pi R$

$\frac{dA}{dt} = \frac{dA}{dR} \cdot \frac{dR}{dt} = (2\pi R)(2)$

$\frac{dA}{dt} = 4\pi R$

$\frac{dA}{dt}(r=3) = 4\pi(3) = 12\pi$ cm²/s

When solving problems with related rates of change, these steps should prove useful.

1. Write down the rate you require, e.g. $\frac{dA}{dt}$.
2. Write down the rate you are given, e.g. $\frac{dr}{dt}$.
3. Use the Chain Rule to link these rates: $\frac{dA}{dt} = \frac{dA}{?} \cdot \frac{?}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$
4. Find an equation connecting the variables of the missing rate, i.e. $A = \pi r^2$.

Example 2

The volume, V cm^3, of water in a container is given by the expression $V = 12h^2$, where h cm is the depth of the water.
Water is flowing into the container at a steady rate of 90 cm^3/sec.
Find the rate, in cm/sec, at which the depth of the water is increasing when $h = 3$.

$V = 12h^2 \Rightarrow \frac{dV}{dh} = 24h$

$\frac{dV}{dt} = 90$ cm^3/s

$\left.\frac{dh}{dt}\right|_{h=3} = ?$

$\frac{dh}{dt} = \frac{dh}{dV} \cdot \frac{dV}{dt}$

$= \left(\frac{1}{24h}\right)\left(\frac{90}{}\right) = \frac{15}{4h}$ cm/s

$h=3$

$\frac{dh}{dt} = \frac{15}{4(3)} \Rightarrow \left.\frac{dh}{dt}\right|_{h=3} = \frac{5}{4}$ cm/s

Exercise 3.6

1. In each of the following, fill in the missing rate:

(i) $\frac{dA}{dt} = \frac{dA}{dr} \cdot \_\_$ (ii) $\frac{dV}{dr} = \frac{dV}{dt} \cdot \_\_$ (iii) $\frac{dM}{dt} = \frac{dM}{ds} \cdot \_\_$

Parametric Differentiation

i $\frac{dA}{dt} = \frac{dA}{dr} \cdot \frac{dr}{dt}$

ii $\frac{dV}{dr} = \frac{dV}{dt} \cdot \frac{dt}{dr}$

iii $\frac{dM}{dt} = \frac{dM}{ds} \cdot \frac{ds}{dt}$

2. In each of the following, find the indicated rate.

(i) $\frac{dA}{dt} = 8$, $\frac{dA}{dr} = 4$, $\frac{dr}{dt} = ?$ (ii) $\frac{dV}{dt} = 8$, $\frac{dr}{dt} = 2$, $\frac{dV}{dr} = ?$

(i) $\frac{dA}{dt} = 8$, $\frac{dA}{dr} = 4$, $\frac{dr}{dt} = ?$

$\frac{dr}{dA} = \frac{1}{4}$

$\frac{dr}{dt} = \frac{dr}{dA} \cdot \frac{dA}{dt} = \left(\frac{1}{4}\right)(8) = 2$

(ii) $\frac{dV}{dr} = \frac{dV}{dt} \cdot \frac{dt}{dr} = (8)(\tfrac{1}{2}) = 4$

$\frac{dt}{dr} = \frac{1}{2}$

3. If $\frac{dy}{dx} = 10$ and $\frac{dx}{dt} = 2$, find $\frac{dy}{dt}$.

$\frac{dy}{dx} = 10$

$\frac{dx}{dt} = 2$

$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$

$= (10)(2)$

$= 20$