Differential Calculus

chapter 2

Section 2.9 Differentiating logarithmic functions

PROJECT MATHS Text & Tests 7

86

When dealing with natural logarithms, the following results are important:

$\log_e e^x = x \log_e e = x$

$\ln e^x = x$, for all $x \in R$

It should be further noted that

(i) $\log_e 1 = 0$ ... the log of 1 to any base is zero

(ii) $\log_e e = 1$ ... $(\log_k k = 1)$

The work involved in differentiating logarithmic functions can be simplified by using the laws of logarithms which are reproduced below:

Laws of Logarithms

(i) $\log_e(xy) = \log_e x + \log_e y$ (ii) $\log_e\left(\frac{x}{y}\right) = \log_e x - \log_e y$

(iii) $\log_e x^n = n \log_e x$ (iv) $\log_a x = \frac{\log_e x}{\log_e a}$

Note: If $y = \log_e(6x)$, we use logarithmic differentiation and the *Chain Rule* to find $\frac{dy}{dx}$.

$$y = \log_e(6x) \Rightarrow \frac{dy}{dx} = \frac{1}{6x} \cdot \frac{d}{dx}(6x) = \frac{1}{6x} \cdot \frac{6}{1} = \frac{1}{x}$$

In general, if $y = \log_e(f(x))$, $\frac{dy}{dx} = \frac{1}{f(x)} \cdot f'(x)$

If $y = \ln x$, then $\frac{dy}{dx} = \frac{1}{x}$.

Example 1

Find $\frac{dy}{dx}$ if (i) $y = \log_e(4x^2 + 1)$ (ii) $y = \log_e(\sin^2 x)$.

$f(x) \rightarrow f'(x)$

$\ln x \rightarrow \frac{1}{x}$

(i) $y = \ln(4x^2+1)$

CHAIN RULE

$$\frac{dy}{dx} = \underbrace{\frac{1}{(4x^2+1)}}\cdot\underbrace{(8x)}$$

$$= \frac{8x}{4x^2+1}$$

3 layer chain rule

(ii) $y = \ln[(\sin x)^2]$

$$\frac{dy}{dx} = \underbrace{\left(\frac{1}{\sin^{\cancel{2}} x}\right)}\cdot\underbrace{(2\cancel{\sin x})}\cdot\underbrace{(\cos x)}$$

$$= \frac{2\cos x}{\sin x} \quad (\text{or } 2\cot x)$$

Example 2

Given that $y = \log_e\left(\frac{1+x}{1-x}\right)$, show that $(1 - x^2)\frac{dy}{dx} = 2$.

$\log \frac{a}{b} = \log a - \log b$

rewrite to separate

$y = \ln(1+x) - \ln(1-x)$

$f(x) \rightarrow f'(x)$ CHAIN RULE

$\ln x \rightarrow \frac{1}{x}$

$$\frac{dy}{dx} = \underbrace{\left(\frac{1}{1+x}\right)}\cdot\underbrace{(1)} - \left(\frac{1}{1-x}\right)(1)$$

Write as single fraction

$$= \frac{1}{(1+x)} - \frac{1}{(1-x)}$$

BOWTIE!

$$= \frac{1(1-x) - 1(1+x)}{(1+x)(1-x)}$$

leave denominator in factorised form

$$= \frac{\cancel{1} - x - \cancel{1} - x}{(1+x)(1-x)}$$

$$= \frac{-x^2}{(1+x)(1-x)}$$