Example 3

Find $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$ given   (i) $f(x)=4x-5$ (ii) $f(x)=x^2+1$.

(i) $\quad f(x)=4x-5$
$\quad f(x+h)=4(x+h)-5$
$\qquad\qquad = 4x+4h-5$
$\quad \dfrac{f(x+h)-f(x)}{h}=\dfrac{4h}{h}=4$

$\quad \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}=4$

(ii) $\quad f(x)=x^2+1$
$\quad f(x+h)=(x+h)^2+1$
$\qquad\qquad = x^2+2xh+h^2+1$
$\quad \dfrac{f(x+h)-f(x)}{h}=\dfrac{2xh+h^2}{h}=2x+h$

$\quad \lim\limits_{x\to 0}\dfrac{f(x+h)-f(x)}{h}=2x+0=2x$

Continuity

A function $f(x)$ is said to be **continuous** when $x=a$ if the graph of $y=f(x)$ can be drawn through the point with coordinates $(a, f(a))$ without a break.
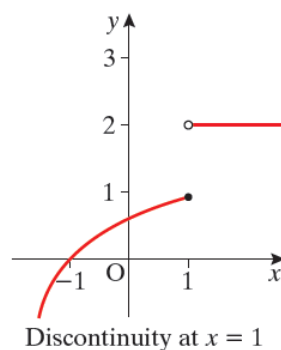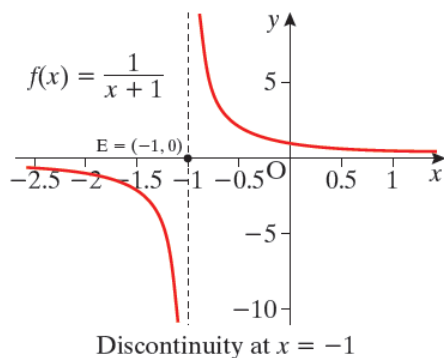Otherwise, there is said to be a **discontinuity** at $x=a$.

Most of the functions that we encounter on our course are continuous functions.

The graph on the right is continuous as the graph can be traced with no jumps or breaks, that is, the pen does not have to leave the page.
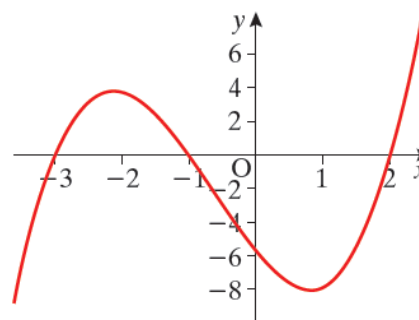
Consider the two graphs shown below:

$f(x) = \dfrac{1}{x+1}$

$E = (-1, 0)$

Discontinuity at $x = -1$

Discontinuity at $x = 1$

On the left above is the graph of the function $f(x) = \dfrac{1}{x+1}$.

As x approaches -1 from below, the value of the function decreases rapidly.
As x approaches -1 from above, the value of the function increases rapidly.

$f(-1)$ cannot be found as $f(-1) = \dfrac{1}{-1+1} = \dfrac{1}{0}$, which is undefined.

We say that the graph is not continuous at $x = -1$.
Notice that the curve approaches but never touches the line $x = -1$.
We say that the line $x = -1$ is an **asymptote** to the curve.

Similarly, the graph on the right above shows a break at $x = 1$.

A function f is continuous at $x = a$ if $\lim\limits_{x \to a} f(x) = f(a)$.

Example 4

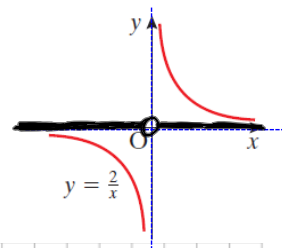Show that $f(x) = \dfrac{3}{x-3}$ is not continuous at $x = 3$.

Value of $f(3) = ?$

$f(3) = \dfrac{3}{3-3} = \dfrac{3}{0} \Rightarrow$ doesn't exist

$\Rightarrow$ not continuous

2. A sketch of the function $y = \frac{2}{x}$ is shown.

 (i) For what value of x is the function discontinuous?

 (ii) Use limits to explain why the function is discontinuous at this point.

$y = \frac{2}{x}$

$x \neq 0$

discontinuous at $x = 0$

$f(0)$ doesn't exist

$\lim\limits_{x \to 0} \dfrac{2}{x} = ?$ undefined.

Approach from left

eg $x = -0.01$, $f(x) = \dfrac{2}{-0.01}$ = neg. large

$\lim\limits_{x \to 0} f(x)$ doesn't exist.

Approach from Right

eg.. $x = 0.01$, $f(x) = \dfrac{2}{0.01}$ = large $\oplus$

$\Rightarrow$ limit does not exist at $x = 0$