Functions

chapter 1

Section 1.5 Limits – Continuity

PROJECT MATHS
Text & Tests 7

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Introduction to limits

Consider this sequence of numbers: $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \ldots$

If we add the first two numbers, we get $\frac{3}{4}$. If we add the first three, we get $\frac{7}{8}$.

If we add the first six numbers, we get $\frac{63}{64}$.

Notice that the more terms we add, the closer the result gets to 1 but it **never reaches** 1.
In mathematics, we say that the **limit** of the sum of these numbers is 1.

We will now take the function $f(x) = x^2$ and consider its value as x **approaches** 3 from below and above 3

① $f(2) = 4$; $f(2.5) = 6.25$; $f(2.75) = 7.5625$; $f(2.9) = 8.41 \ldots$
② $f(4) = 16$; $f(3.5) = 12.25$; $f(3.25) = 10.5625$; $f(3.1) = 9.61 \ldots$

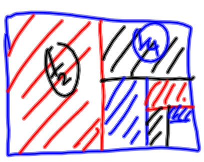
As x gets closer to 3, the value of x^2 gets closer to 9.
9 is said to be the limit of $f(x) = x^2$ as x **tends to** 3.

This is written as $\lim\limits_{x \to 3} (x^2) = 9$.

Limit notation

$\lim\limits_{x \to a} f(x) = p$

states that $f(x)$ approaches p as x gets close to a.



we see total < 1

In general, to find the limit of $\lim_{x \to a} f(x)$, we substitute a for x in the function.

For example, $\lim_{x \to 2} \frac{3x+2}{x+4} = \frac{3(2)+2}{2+4} = \frac{8}{6} = \frac{4}{3}$.

Now consider $\lim_{x \to 3} \frac{x^2-9}{x-3}$.

When we substitute 3 for x, we get $\frac{9-9}{3-3} = \frac{0}{0}$.

The result $\frac{\mathbf{0}}{\mathbf{0}}$ is known as an **indeterminate form** as its value cannot be determined.

If after substitution the result is $\frac{0}{0}$, some other method must be found to obtain the limit.

The most common method used involves factorising the numerator and denominator and then dividing by the common factor.

Thus, $\lim_{x \to 3} \frac{x^2-9}{x-3} = \lim_{x \to 3} \frac{(x+3)\cancel{(x-3)}^{1}}{\cancel{(x-3)}_{1}} = \lim_{x \to 3}(x+3) = 6$.

Note: It is important to realise that if $f(x) = \frac{x^2-9}{x-3}$, then $f(3)$ does not exist, but the $\lim_{x \to 3}(x)$ does exist.

Example 1

Evaluate (i) $\lim_{x \to 0} \frac{4x+1}{2x+3}$ (ii) $\lim_{x \to 2} \frac{x^2+x-6}{x-2}$.

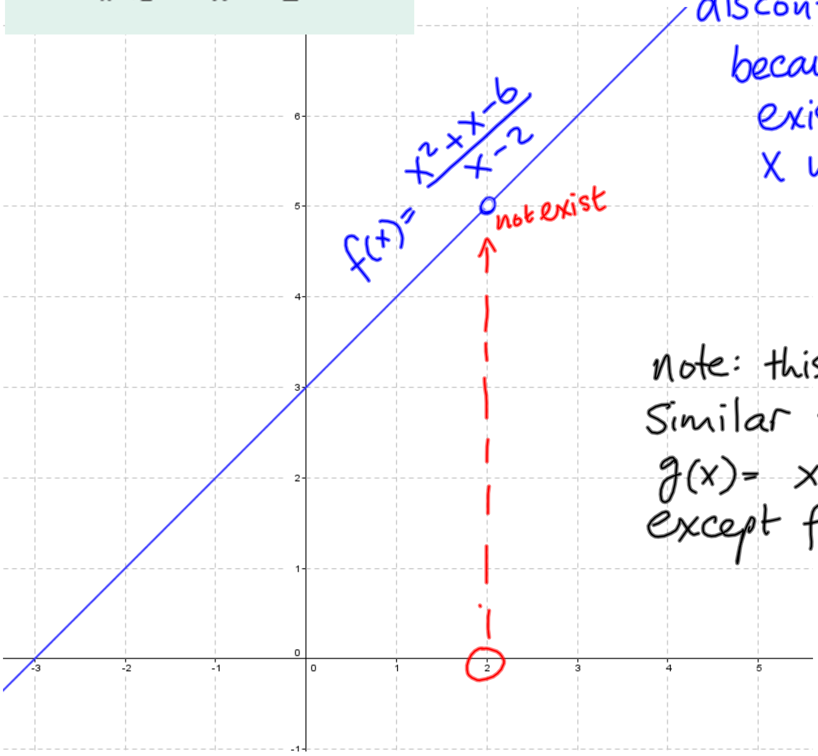
(i) $\lim_{x \to 0} \frac{4(0)+1}{2(0)+3} = \frac{1}{3}$

(ii) $\lim_{x \to 2} \frac{x^2+x-6}{x-2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+3)}{\cancel{x-2}}$

can't sub in 2 because f(2) doesn't exist!

$= \lim_{x \to 2} (x+3) = 2+3 = 5$

(ii) $\lim\limits_{x \to 2} \dfrac{x^2 + x - 6}{x - 2}$.



discontinuous because it doesn't exist for all x values.

note: this graph is similar shape to $g(x) = x - 3$ except for discontinuity

Limit of a function as $x \to \infty$

In the example above, we investigated the limit of a function as the variable tended to a fixed number. Now we will examine the limit of a function as the variable tends to infinity. We use the symbol ∞ to denote infinity.

Now consider $\lim\limits_{x \to \infty} \left(\dfrac{1}{x}\right)$.

When $x = 10, \dfrac{1}{x} = \dfrac{1}{10} = 0.1$. When $x = 1000, \dfrac{1}{x} = \dfrac{1}{1000} = 0.001$.

When $x = 1{,}000{,}000, \dfrac{1}{x} = \dfrac{1}{1{,}000{,}000} = 0.000001$.

These examples illustrate that as x increases, the value of $\dfrac{1}{x}$ decreases and in fact tends to zero.

$\lim\limits_{x \to \infty} \dfrac{1}{x} = 0$

If the numerator is any fixed number k, the $\lim\limits_{x \to \infty} \dfrac{k}{x}$ is also zero.

Since $\lim\limits_{x \to \infty} \dfrac{1}{x} = 0$, it follows that $\lim\limits_{x \to \infty} \dfrac{1}{x^2} = 0$ and $\lim\limits_{x \to \infty} \dfrac{k}{x^2} = 0$, where $k \in R$.

$\boxed{\dfrac{1}{\infty} \approx 0}$ $\boxed{\dfrac{1}{\infty^2} \approx 0}$

Example 2

Evaluate (i) $\lim_{x\to\infty} \frac{4x+1}{2x+3}$ (ii) $\lim_{x\to\infty} \frac{3x^2-2x+4}{5x^2+4x-3}$

Trick multiply by $\frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}$

$\lim_{x\to\infty} \frac{1}{x} = 0$

$\lim_{x\to\infty} \frac{3}{x} = 0$

(i) $= \lim_{x\to\infty} \frac{\frac{1}{x}(4x+1)}{\frac{1}{x}(2x+3)}$

$= \lim_{x\to\infty} \frac{4+\frac{1}{x}}{2+3/x} = \frac{4+0}{2+0}$

$= \frac{4}{2} = 2$

Trick multiply by: $\frac{\frac{1}{x^2}}{\frac{1}{x^2}}$

(ii) $= \lim_{x\to\infty} \frac{\frac{1}{x^2}(3x^2-2x+4)}{\frac{1}{x^2}(5x^2+4x-3)}$

$= \lim_{x\to\infty} \frac{3-2/x+4/x^2}{5+4/x-3/x^2}$

$= \frac{3-0+0}{5+0-0} = \frac{3}{5}$