Functions

Chapter 1

Section 1.4 Inverse functions

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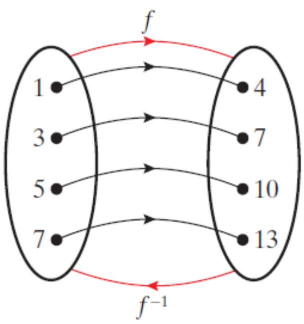
PROJECT MATHS

Text & Tests 7

In previous sections, we have seen that a function generates a set of y-values, called the range, from a set of x-values, called the domain.



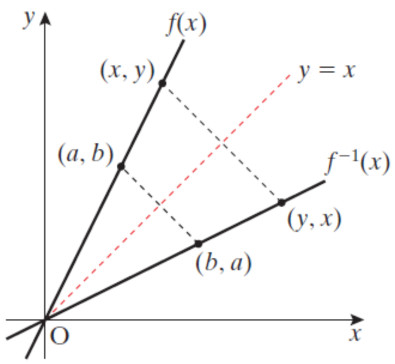
In this section, we will deal with the reverse of this procedure by finding the elements of the domain when given the elements of the range.

In the given mapping diagram, the couples of f are:

$$f = (1, 4), (3, 7), (5, 10), (7, 13)$$

If we reverse these couples, we get a new function called f^{-1}.
f^{-1} is said to be the **inverse function** of f.
From the mapping diagram, $f^{-1} = (4, 1), (7, 3), (10, 5), (13, 7)$.

For every couple (a, b) that f creates, f^{-1} will create the couple (b, a).
Thus, f and f^{-1} create points that are reflections of one another in the line $y = x$.



Note: As can be seen from the diagram, a function f has an inverse if and only if it is bijective, i.e. the function must be one-to-one.
Take the function $f(x) = x^2$.
$(2, 4)$ and $(-2, 4)$ are couples of f.
So the couples $(4, 2)$ and $(4, -2)$ are couples of f^{-1}.
This shows that f^{-1} is not a function as the input 4 does not have a **unique** output.
Since $f(x) = x^2$ is not bijective, it does not have an inverse.

How to find the inverse of a function

Let $f(x) = 3x - 2$

$$\Rightarrow \quad y = 3x - 2$$

$$3x = y + 2$$

$$x = \frac{y+2}{3} \quad \text{... express } x \text{ in terms of } y$$

$$\therefore \quad f^{-1}(x) = \frac{x+2}{3} \quad \text{... replace } y \text{ with } x$$

$f^{-1}(x) = \dfrac{x+2}{3}$ is the inverse function of $f(x)$.

We can verify that the inverse function is correct by showing that $f(3) = 7$ and $f^{-1}(7) = 3$.

Example 1

(i) If $f(x) = 5x - 3$, find $f^{-1}(x)$.

(ii) Hence, show that $f^{-1}f(x) = x$.

Steps

(1) replace $f(x)$ with y

(2) rearrange so we have x in terms of y

(3) replace x with $f^{-1}(x)$ and y with x

$$f(x) = 5x - 3$$
$$y = 5x - 3$$
$$y + 3 = 5x$$
$$x = \frac{y+3}{5}$$

inverse function

$$\boxed{f^{-1}(x) = \frac{x+3}{5}}$$

$f^{-1}[f(x)] = ?$

$$f^{-1}(5x-3) = \frac{(5x-3)+3}{5}$$
$$= \frac{5x}{5} = x$$

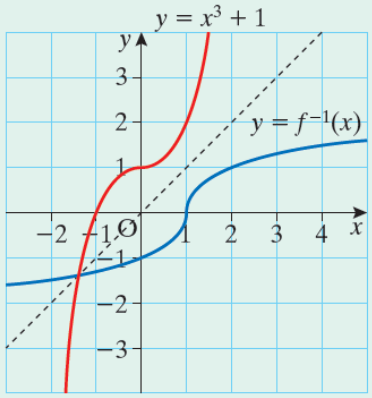
QED

In Example 1 above, it was shown that $f^{-1}f(x) = x$.
It can also be shown that $ff^{-1}(x) = x$.

For any two functions f and f^{-1}, then
(i) $ff^{-1}(x) = x$.
(ii) $f^{-1}f(x) = x$.

Example 2

The diagram on the right shows the reflection of the curve $f(x) = x^3 + 1$ in the line $y = x$.
Find the equation of $f^{-1}(x)$.



replace $f(x)$ with y

$$y = x^3 + 1$$

rearrange to have x in terms of y

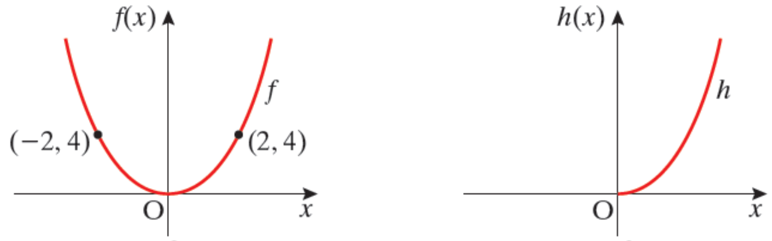
$$x^3 = y - 1$$
$$x = \sqrt[3]{y-1}$$

replace x with $f^{-1}(x)$ and y with x

$$f^{-1}(x) = \sqrt[3]{x-1}$$

Restricted domain

Consider the two graphs shown below:



$f(x) = x^2, x \in R$ $\qquad$ $h(x) = x^2, x \geqslant 0$

$f(x) = x^2, x \in R$ is a function but it is not injective as a horizontal line will intersect the graph more than once. Since it is not bijective, it has no inverse.
$h(x) = x^2, x \geqslant 0$ is a function in the restricted domain $x \geqslant 0$.
$h(x)$ is a bijective function and hence has an inverse.
The inverse function $h^{-1}(x)$ is $y = \sqrt{x}$, for $x \geqslant 0$.
If f and f^{-1} are inverse functions, the domain of f is the range of f^{-1}.

Example 3

The relation $f(x) = x^2 - 4x - 4$ is a function with domain $x > 2$.
Find $f^{-1}(x)$ and write down its range.

replace $f(x)$ with y

$y = x^2 - 4x - 4$

rearrange $x = ?$
need to write in complete square form

$= x^2 - 4x + 4 - 4 - 4$

$y = (x-2)^2 - 8$

	x	-2
x	x^2	$-2x$
-2	$-2x$	$+4$

$(x-2)^2 = x^2 - 4x + 4$

$(x-2)^2 = y + 8$

$x - 2 = \sqrt{y+8}$

$x = \sqrt{y+8} + 2$

Inverse function

Swap x with $f^{-1}(x)$ and y with x

$f^{-1}(x) = \sqrt{x+8} + 2$

Range in $f^{-1}(x)$ = domain in $f(x)$

range > 2 i.e. $(2, \infty)$