Algebra 1

chapter 1

Key words

polynomial **expression** **equation** **variable** **linear** **quadratic** **cubic**
expanding **degree** **factors** **identity** **in terms of** **simultaneous**

Section 1.1 Polynomial expressions

Polynomial expressions are formed from the addition of **many** algebraic terms with **positive integer powers.**

$5x^3 - 3x^2 + 4x - 6$ is a polynomial expression.

This expression consists of four **terms**;
$5x^3, -3x^2, 4x$ and -6.

The **degree** of a polynomial is given by the highest power of x.

A **monomial** expression contains one term.

A **binomial** expression contains two terms.

A **trinomial** expression contains three terms.

(a) $4x - 6$ is a **linear** polynomial since the highest power (degree) of x is 1.
(b) $-3x^2 + 4x - 6$ is a **quadratic** polynomial of degree 2.
(c) $5x^3 - 3x^2 + 4x - 6$ is a **cubic** polynomial of degree 3.

Each polynomial is written in order of (i) decreasing powers of x, e.g. $5x^3 - 3x^2 + 4x - 6$,
or (ii) increasing powers of x, e.g. $9 + 3x - 4x^2$.

The **coefficient** of x^3 is the number before x^3 in the expression.

In $4x^3 - 2x^2 + 5x - 6$, the coefficient of x^3 is 4, the coefficient of x^2 is -2, the coefficient of x is 5, and -6 is the **constant term**.

1. Addition and subtraction of polynomial expressions

Each term in a polynomial represents a different quantity, e.g. $8, 6x, 4x^2$.

When simplifying an expression, all like terms should be combined into a single term.

Example 1

Expand and simplify each of the following expressions.

(i) $7(x^3 + 2x^2 - 5x) - 2(2 + 3x + 4x^2 - 2x^3)$

(ii) $3x^2(4x^2 - 5x + 6) + 4x(8x^3 - 2x - 3)$

(i) $7(x^3 + 2x^2 - 5x) - 2(2 + 3x + 4x^2 - 2x^3) = 7x^3 + 14x^2 - 35x - 4 - 6x - 8x^2 + 4x^3$
$= 11x^3 + 6x^2 - 41x - 4$

(ii) $3x^2(4x^2 - 5x + 6) + 4x(8x^3 - 2x - 3) = 12x^4 - 15x^3 + 18x^2 + 32x^4 - 8x^2 - 12x$
$= 44x^4 - 15x^3 + 10x^2 - 12x$

2. Multiplying polynomial expressions

To multiply algebraic expressions, we use the distributive law, i.e. $a(b + c) = ab + ac$.

Example 2

Simplify the following: $(x - 5)(2x^2 - 3x + 6)$

$$\begin{aligned}(x - 5)(2x^2 - 3x + 6) &= x(2x^2 - 3x + 6) - 5(2x^2 - 3x + 6)\\ &= 2x^3 - 3x^2 + 6x - 10x^2 + 15x - 30\\ &= 2x^3 - 13x^2 + 21x - 30\end{aligned}$$

Note: Multiplying polynomial expressions is often called **expanding**.

3. Perfect squares

Any polynomial of the form $(x + a)^2$ is called a perfect square.

$$\begin{aligned}(x + a)^2 &= (x + a)(x + a)\\ &= (x)(x + a) + (a)(x + a)\\ &= x^2 + ax + ax + a^2\\ &= x^2 + 2ax + a^2\end{aligned}$$

Similarly, $(x - a)^2 = x^2 - 2ax + a^2$.

$(x + a)^2 = x^2 + 2ax + a^2$
$(x - a)^2 = x^2 - 2ax + a^2$

For example, $(2x - 3)^2 = (2x)^2 - 2(2x)(3) + (3)^2 = 4x^2 - 12x + 9$

Example 3

Given that $25x^2 + px + 16$ is a perfect square and $p > 0$, find the value of p.

Since $25x^2 = (5x)^2$ and $16 = (4)^2$, $\therefore$ $25x^2 + px + 16 = (5x + 4)^2$

$\therefore$ $2(5x)(4) = px$

$\therefore$ $40x = px \Rightarrow p = 40$.

(**Note:** $16 = (-4)^2$ $\therefore$ $2(5x)(-4) = px \Rightarrow p = -40$, which is not valid because $p > 0$)

4. Expanding $(x - a)(x + a)$

The expansion of $(x - a)(x + a) = x^2 + ax - ax - a^2$
$$= x^2 - a^2$$

$$(x - a)(x + a) = x^2 - a^2$$

Similarly, $(a - 5b)(a + 5b) = a^2 - 5ab + 5ab - (5b)^2 = \mathbf{a^2 - (5b)^2}$
$$= a^2 - 25b^2$$

This expansion results in a binomial expression called the **difference of two squares.**
This result is important when we need to factorise an expression of the form $a^2 - b^2$ as we will see later on in the chapter.

5. Dividing algebraic expressions

Algebraic quotients take many different forms. Some may be simplified as in the following cases.

Case 1. The denominator is a factor of each term of the numerator.

(i) $$\frac{6x^3 - 8x^2y + 4xy^2 + 2x^2}{2x} = \frac{6x^3}{2x} - \frac{8x^2y}{2x} + \frac{4xy^2}{2x} + \frac{2x^2}{2x} = 3x^2 - 4xy + 2y^2 + x$$

Case 2. The denominator is one of the factors of the numerator.

(ii) $$\frac{6x^2 + 5xy + y^2}{(2x + y)} = \frac{(3x + y)(2x + y)}{(2x + y)} = 3x + y$$

Case 3. The denominator divides into the numerator using long division.

(iii) $$\frac{2x^3 - 9x^2 + 10x - 3}{(x - 3)} = 2x^2 - 3x + 1$$ using long division.

Long Division

$$\begin{array}{rll}
 & 2x^2 - 3x + 1 & \\
x - 3 \,\big) & 2x^3 - 9x^2 + 10x - 3 & \text{… divide } 2x^3 \text{ by } x \text{ to get } 2x^2 \\
 & 2x^3 - 6x^2 \quad \text{(subtract)} & \text{… multiply } 2x^2 \text{ by denominator} \\
 & -3x^2 + 10x - 3 & \text{… divide } -3x^2 \text{ by } x \text{ to get } -3x \\
 & -3x^2 + 9x \quad \text{(subtract)} & \text{… multiply } -3x \text{ by denominator} \\
 & x - 3 & \text{… divide } x \text{ by } x \text{ to get } 1 \\
 & x - 3 \quad \text{(subtract)} &
\end{array}$$

Hence, $$\frac{2x^3 - 9x^2 + 10x - 3}{(x - 3)} = 2x^2 - 3x + 1.$$

Example 4

Divide $(2x^3 - 11x + 6)$ by $(2x^2 + 4x - 3)$.

Since this cubic polynomial has no power of x^2, it is good practice to rewrite the polynomial leaving space for the x^2 coefficients as follows;

$$\begin{array}{rll}
 & x - 2 & \\
2x^2 + 4x - 3 & \overline{)\,2x^3 \qquad\qquad - 11x + 6} & \text{... divide } 2x^2 \text{ into } 2x^3 \text{ to get } x \\
 & \underline{2x^3 + 4x^2 - 3x} & \text{... multiply } x \text{ by } 2x^2 + 4x - 3 \text{ and then subtract} \\
 & \qquad -4x^2 - 8x + 6 & \text{... divide } 2x^2 \text{ into } -4x^2 \text{ to get } -2 \\
 & \qquad \underline{-4x^2 - 8x + 6} & \text{... multiply } -2 \text{ by } 2x^2 + 4x - 3 \text{ and then subtract}
\end{array}$$

$\therefore \quad (2x^3 - 11x + 6) \div (2x^2 + 4x - 3) = x - 2$

Note also that when we divide $2x^3 - 11x + 6$ by $x - 2$ we get $2x^2 + 4x - 3$.

$$\begin{array}{rl}
 & 2x^2 + 4x - 3 \\
x - 2 & \overline{)\,2x^3 \qquad\quad - 11x + 6} \\
 & \underline{2x^3 - 4x^2} \\
 & \qquad 4x^2 - 11x + 6 \\
 & \qquad \underline{4x^2 - 8x} \\
 & \qquad\qquad -3x + 6 \\
 & \qquad\qquad \underline{-3x + 6}
\end{array}$$

The factors of the polynomial $2x^3 - 11x + 6$ are $(x - 2)$ and $(2x^2 + 4x - 3)$,

i.e. $\quad 2x^3 - 11x + 6 = (x - 2)(2x^2 + 4x - 3)$.

We will use this property more fully in the chapter on factorisation.

Exercise 1.1

1. Given the polynomial $4x^3 + 3x^2 - 9x + 5$, write down

(i) the coefficient of x^2
(ii) the coefficient of x
(iii) the term independent of x (the constant term).

2. State the degree of each of the following polynomial expressions.

(i) $-3x^2 + 5x - 1$ (ii) $4x^3 - 4x^2 + 9x + 3$ (iii) $7 + 3x - 3x^3 - 6x^4$

3. Give two reasons why $3x^2 - \dfrac{4}{x} + x^{\frac{3}{2}}$ is not a polynomial.

4. Simplify each of the following.

(i) $3x^2 - 6x + 7 + 5x^2 + 2x - 9$ (ii) $x^3 - 4x^2 - 5x + 3x^3 + 6x^2 - x$
(iii) $x(x + 4) + 3x(2x - 3)$ (iv) $3(x^2 - 7) + 2x(3x - 1) - 7x + 2$

5. Simplify each of the following.

(i) $3x^2(4x + 2) + 5x^2(2x - 5)$ (ii) $x^3(x - 2) + 4x^3(2x - 6)$
(iii) $x(x^3 + 4x^2 - 7x) + 3x^2(2x^2 - 3x + 4)$ (iv) $3x(x^2 - 7x + 1) + 2x^2(6x - 5)$

6. Expand each of the following.

(i) $(x + 4)(2x + 5)$ (ii) $(2x + 3)(x - 2)$ (iii) $(3x - 2)(x + 3)$
(iv) $(3x - 2)(4x - 1)$ (v) $(3x - 1)(2x + 5)$ (vi) $(4x + 1)(2x - 6)$
(vii) $(x - 2)(x + 2)$ (viii) $(2x + 5)(2x - 5)$ (ix) $(ax - by)(ax + by)$

7. Expand each of the following perfect squares.

(i) $(x + 2)^2$ (ii) $(x - 3)^2$ (iii) $(x + 5)^2$
(iv) $(a + b)^2$ (v) $(x - y)^2$ (vi) $(a + 2b)^2$
(vii) $(3x - y)^2$ (viii) $(x - 5y)^2$ (ix) $(2x + 3y)^2$

8. Express each of the following in the form $ax^2 + bx + c$.

(i) $(x + \frac{1}{2})^2$ (ii) $8(x - \frac{1}{4})^2$ (iii) $-(1 - x)^2$

9. Which of the following are perfect squares? Explain your answers.

(i) $x^2 + 5x + 25$ (ii) $9x^2 - 6x - 1$ (iii) $4 + 12x + 9x^2$

10. If $px^2 + 4x + 1$ is a perfect square for all values of x, find the value of p.

11. If $25x^2 + tx + 4$ is a perfect square for all values of x, find the value of t.

12. If $9x^2 + 24x + s$ is a perfect square for all values of x, find the value of s.

13. Expand and simplify each of the following.

(i) $(x + 2)(x^2 + 2x + 6)$ (ii) $(x - 4)(2x^2 + 3x - 1)$
(iii) $(2x + 3)(x^2 - 3x + 2)$ (iv) $(3x - 2)(2x^2 - 4x + 2)$

14. Show that $(x + y)(x^2 - xy + y^2) = x^3 + y^3$.

15. Verify that $(x - y)(x^2 + xy + y^2) = x^3 - y^3$.

16. Find the coefficient of x in the expansion of $(2x - 3)(3x^2 - 2x + 4)$.

17. Expand fully and simplify $(x + 3)(x - 4)(2x + 1)$.

18. Expand fully and simplify $(x^2 - 3x - 2)(2x^2 - 4x + 1)$.

19. Find the coefficient of x^2 in the expansion of $(3x^2 + 5x - 1)(2x^2 - 6x - 5)$.

20. Simplify each of the following quotients:

(i) $\frac{3x + 6}{3}$ (ii) $\frac{x^2 + 2x}{x}$ (iii) $\frac{3x^3 - 6x^2}{3x}$ (iv) $\frac{15x^2y - 10xy^2}{5xy}$

21. Simplify each of the following quotients:

(i) $\frac{6x^2y + 9xy^2 - 3xy}{3xy}$ (ii) $\frac{6x^4 - 9x^3 + 12x^2}{3x^2}$

22. Simplify each of the following:

(i) $\dfrac{12a^2b}{3ab}$ (ii) $\dfrac{12a^2bc}{3ac}$ (iii) $\dfrac{4xy^2z}{2xy}$ (iv) $\dfrac{3xy}{2} \times \dfrac{4}{6x^2}$

23. Simplify each of the following:

(i) $\dfrac{2x^2 + 5x - 3}{2x - 1}$ (ii) $\dfrac{2x^2 - 2x - 12}{x - 3}$ (iii) $\dfrac{8x^2 + 8x - 6}{4x - 2}$

24. Divide each of the following:

(i) $x^3 - 8x^2 + 19x - 12 \div (x - 1)$
(ii) $2x^3 - x^2 - 2x + 1 \div (2x - 1)$
(iii) $3x^3 - 4x^2 - 3x + 4 \div (3x - 4)$
(iv) $4x^3 - 7x^2 - 21x + 18 \div (x - 3)$
(v) $x^3 - 22x + 15 \div (x + 5)$
(vi) $2x^3 - x^2 - 12 \div (x - 2)$

25. Perform the following operations:

(i) $x^3 - 2x^2 + 2x - 4 \div (x^2 + 2)$
(ii) $x^3 - 9x^2 + 27x - 27 \div (x^2 - 6x + 9)$
(iii) $3x^3 + 2x^2 - 7x + 2 \div (x^2 + x - 2)$
(iv) $5x^3 + 14x^2 + 7x - 2 \div (5x^2 + 4x - 1)$

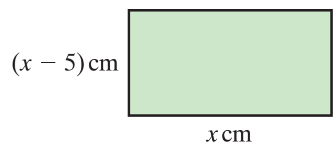
26. Divide each of these:

(i) $x^3 - 8 \div (x - 2)$
(ii) $8x^3 - 27y^3 \div (2x - 3y)$

Section 1.2 Polynomial functions, an introduction

Polynomial functions arise as we try to solve day-to-day problems.

Let x cm be the length of a rectangle.



If the width of the rectangle is 5 cm shorter than the length, then $(x - 5)$ cm is the width.

The area, A, of the rectangle depends on the length and width and by extension depends on x.

The symbol for the area depending on x is written as $\boldsymbol{A(x)}$.

Therefore, $A(x) = x(x - 5) = x^2 - 5x$. As x varies, the area A varies.

x is called the **independent variable** and $A(x)$ the **dependent variable**.

$A(x)$ is the quadratic polynomial $x^2 - 5x$ of degree 2.

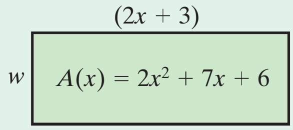
We note that if $x = 10$ cm, then $A(10) = (10)^2 - 5(10) = 50\text{ cm}^2$.

We also note that the width is $(x - 5) \Rightarrow x - 5 > 0$
$\Rightarrow x > 5$ cm

Example 1

The length of a rectangle is $(2x + 3)$ cm. If the area of the rectangle is given by the polynomial function $A(x) = 2x^2 + 7x + 6$, find

(a) an expression for the width of the rectangle
(b) an expression for the perimeter, $P(x)$, of the rectangle
(c) the minimum value of x.



Let w be the width of the rectangle.

(a) Area $A(x) = 2x^2 + 7x + 6 = w(2x + 3)$

$$\therefore \quad w = \frac{2x^2 + 7x + 6}{(2x + 3)} = \frac{(2x + 3)(x + 2)}{(2x + 3)} = (x + 2).$$

(b) The perimeter $P(x) = 2(2x + 3) + 2(x + 2) = 4x + 6 + 2x + 4 = 6x + 10.$

(c) Since $(2x + 3)$ is the length of the rectangle,
$\Rightarrow (2x + 3) > 0$
$\Rightarrow x > -1.5$

Note 1: $A(x)$ must be understood as a single concept and does not imply that A is multiplied by x.
It simply tells us that the quantity A depends on a variable x.

Note 2: Polynomial functions can be added and subtracted as before by collecting like terms and simplifying.

Example 2

Given $f(x) = 3x^{3\,-} 4x^2 - 3x + 4$ and $g(x) = 5x^3 + 14x^2 + 7x - 2$, find

(a) $2f(x) - g(x)$ and state its degree
(b) $f(x) + 2g(x)$ and state its degree.

(a) $2f(x) - g(x) = 2(3x^3 - 4x^2 - 3x + 4) - (5x^3 + 14x^2 + 7x - 2)$
$= 6x^3 - 8x^2 - 6x + 8 - 5x^3 - 14x^2 - 7x + 2$
$= x^3 - 22x^2 - 13x + 10$ which is of degree 3.

(b) $f(x) + 2g(x) = (3x^3 - 4x^2 - 3x + 4) + 2(5x^3 + 14x^2 + 7x - 2)$
$= 3x^3 - 4x^2 - 3x + 4 + 10x^3 + 28x^2 + 14x - 4$
$= 13x^3 + 24x^2 + 11x$ which is of degree 3.

Evaluating polynomial functions

The value of a polynomial function is obtained by substituting a given value for the independent variable and simplifying.

If $p(x) = 2x^2 - 5x + 6$, then $p(1) = 2(1)^2 - 5(1) + 6 = 3$
and $p(-3) = 2(-3)^2 - 5(-3) + 6 = 39.$

A new variable can also be introduced in a similar way.

Given that $p(x) = 2x^2 - 5x + 6$,

$p(t) = 2t^2 - 5t + 6$.

Also $p(t^2) = 2(t^2)^2 - 5(t^2) + 6 = 2t^4 - 5t^2 + 6$

Example 3

A paint manufacturer knows that the daily cost (€C) of producing x litres of paint is given by the formula $C(x) = 0.001x^2 + 0.1x + 5$.

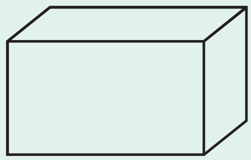
(a) State the degree of $C(x)$.
(b) Find the daily cost of producing (i) 100 ℓ of paint (ii) 400 ℓ of paint.

(a) The degree of $C(x)$ is 2.

(b) (i) $C(100) = 0.001(100)^2 + 0.1(100) + 5 = €25$
(ii) $C(400) = 0.001(400)^2 + 0.1(400) + 5 = €205$.

Example 4

An open box has dimensions $x + 3, x + 1$ and x, where x is the height (in cms) of the box. Find an expression for the external surface area of the box, $S(x)$, and hence find $S(5)$.



Area of the sides $= 2(x)(x + 3) + 2(x)(x + 1)$
$= 2x^2 + 6x + 2x^2 + 2x = 4x^2 + 8x$

Area of the base $= (x + 3)(x + 1) = x^2 + 4x + 3$

Total surface area $S(x) = x^2 + 4x + 3 + 4x^2 + 8x$
$= (5x^2 + 12x + 3)\text{ cm}^2$

$S(5) = 5(5)^2 + 12(5) + 3 = 188\text{ cm}^2$

Example 5

Given the function $f(x) = 2x - 4$ for all $x \in R$, find

(a) $f(3), f(-2), f(t)$
(b) for what values of t is $f(t) = t$.

(a) $f(x) = 2x - 4 \Rightarrow f(3) = 2(3) - 4 = 2$
$f(-2) = 2(-2) - 4 = -8$
$f(t) = 2(t) - 4 = 2t - 4$

(b) $f(t) = t \Rightarrow 2t - 4 = t$
$t - 4 = 0$
$t = 4$

Note: Polynomial functions with more than one independent variable occur regularly.

The volume of a cylinder $V = \pi.r^2.h$, where r is the radius of the base and h the height of the cylinder.

In function terms: $V(r, h) = \pi.r^2.h$,
that is, the volume of the cylinder depends on both the radius, r, and the height, h.

The volume V depends on two independent variables, r, h.
The degree of this polynomial is 2, the highest power (index) of either variable.

Exercise 1.2

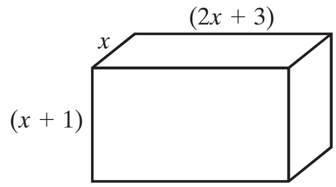
1. A rectangle has one side 4 cm longer than the other.
Let x be the length of the smaller side.

Find (i) an expression for $A(x)$, the area of the rectangle
(ii) an expression for $P(x)$, the perimeter of the rectangle.

2. The area of a rectangle, $A(x)$, is $6x^2 + 4x - 2$.
If the length is given by $(3x - 1)$, find

(i) an expression for the width of the rectangle
(ii) an expression for the perimeter, $P(x)$, of the rectangle.

3. The dimensions (in cm) of an open rectangular box are given in the diagram. Find



(a) an expression for the volume, $V(x)$, of the box
(b) an expression for the external surface area, $S(x)$, of the box
(c) the value of
(i) $V(x)$ and (ii) $S(x)$ when $x = 5$.

4. If $f(x) = 2x^3 - x^2 - 5x - 4$, find

(a) $f(0)$ (b) $f(1)$ (c) $f(-2)$ (d) $f(3a)$

5. If $f(x) = x^2 - 3x + 6$, find

(a) $f(0)$ (b) $f(-5)$ (c) $f\left(-\frac{1}{2}\right)$ (d) $f\left(\frac{a}{4}\right)$

6. A rectangle has length $(x - y)$ and width $(2x + 3y)$.
Find, in terms of x and y, an expression for the

(a) area (b) perimeter of the rectangle.

7. The width of an open rectangular box is 5 cm shorter than its length and the height of the box is twice the length.
By letting the length of the box be x cm, find
(a) an expression for the volume, $V(x)$, of the box
(b) an expression for the total surface area, $S(x)$ (internal and external), of the box.

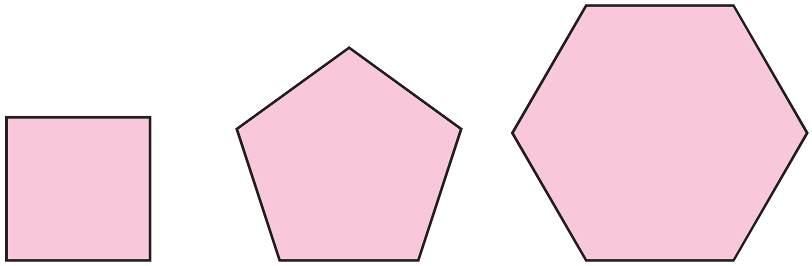
8. The number of diagonals, d, in an n-sided polygon is given by the polynomial

$$d(n) = \frac{n^2}{2} - \frac{3n}{2}.$$

Explain what is meant by (i) $d(4)$ (ii) $d(5)$ and find values for $d(4), d(5), d(6)$.

Copy each polygon below and verify your answer in each case.

Explain why $d(3) = 0$.



9. If $f(x) = x + 5$, find, in terms of a, $f(a^2) - 3f(a) + 2$.

10. Given $f(x) = x^2 - 3x + 6$, find

(i) $f(-2t)$ (ii) $f(t^2)$ (iii) $f(t-2)$

State the degree of each of the polynomial functions in t.

11. The volume of a cone, $V(r, h)$, is given by the formula $V(r, h) = \frac{1}{3}\pi r^2 h$, where r is the radius and h is the perpendicular height of the cone. Find

(i) the volume, in terms of π, of a cone with height 21 cm and radius 14 cm

(ii) the volume of a cone, in terms of r and π, if the cone has the same height as the radius r

(iii) the volume of a cone, in terms of h and π, if the radius of the base is twice the height h.

12. If $f(x) = 3x + 6$, find $f(10)$.

If $f(x) = 2x + 8$, find $f(10)$.

By studying the pattern of the results above, if $g(10) = 47$, write $g(x)$ in the form $ax + b$.

13. Use the formula $T = 2\pi\sqrt{\frac{l}{g}}$ to find the value of l, in terms of π, if $T = 4\,\text{s}$ and $g = 10\,\text{m}\,\text{s}^{-1}$.

14. Use the formula $V = \frac{4}{3}\pi r^3$ to find the value of r if $V = \frac{792}{7}\,\text{m}^3$ and $\pi = \frac{22}{7}$.

15. In the morning, every student in a classroom shakes hands with every other student as a greeting. The number of handshakes, H, between x students is given by the expression $H(x) = \frac{x}{2}(x - 1)$.

Using this formula, find

(i) the number of handshakes between 5 students
(ii) the number of handshakes between 6 students
(iii) the number of handshakes between 10 students.

Using a pattern created by the above, or otherwise, find, if on a particular morning 136 handshakes were given, the number of students in the room.

Section 1.3 Factorising algebraic expressions

An algebraic **factor** divides evenly into a polynomial leaving no remainder.

$(x - 3)$ is a factor of $2x^3 - 9x^2 + 10x - 3$
because $(2x^3 - 9x^2 + 10x - 3) \div (x - 3) = (2x^2 - 3x + 1)$.

$(x - 4)$ and $(x + 3)$ are both factors of $x^2 - x - 12$ because
(i) $(x^2 - x - 12) \div (x - 4) = (x + 3)$ and also (ii) $(x^2 - x - 12) \div (x + 3) = (x - 4)$.

To solve algebraic equations, we first need to be able to factorise different algebraic expressions.

Several different techniques can be used for factorising expressions and these are shown below.

1. Finding the highest common factor by inspection

(i) $3x^2 - 9xy = 3x(x - 3y) \Rightarrow$ the factors are $3x$ and $(x - 3y)$
(ii) $2a^2b - 4ab^2 + 12\,abc = 2ab(a - 2b + 6c) \Rightarrow$ the factors are $2ab$ and $(a - 2ab + 6c)$.

2. Grouping terms

$$\begin{aligned} 6x^2y + 3xy^2 - 12x - 6y &= 3xy(2x + y) - 6(2x + y) \\ &= (2x + y)(3xy - 6) \\ &\Rightarrow \text{the factors are } (2x + y) \text{ and } (3xy - 6) \end{aligned}$$

3. Difference of two squares

Since $(x + y)(x - y) = x^2 - y^2$, the factors of $x^2 - y^2$ are $(x + y)$ and $(x - y)$.

Note: When simplifying quotients, it is important to be able to factorise fully expressions containing the difference of two squares.

(i) $c^2 - d^2 = (c - d)(c + d)$
(ii) $x^2 - 9y^2 = x^2 - (3y)^2 = (x - 3y)(x + 3y)$
(iii) $x^2 - 8y^2 = x^2 - (\sqrt{8}y)^2 = (x - \sqrt{8}y)(x + \sqrt{8}y)$
(iv) $x - 9 = (\sqrt{x})^2 - 3^2 = (\sqrt{x} - 3)(\sqrt{x} + 3)$

Example 1

Factorise fully (i) $x^4 - y^4$ (ii) $12x^2 - 75y^2$

(i) $x^4 - y^4 = (x^2)^2 - (y^2)^2$ … write as the difference of two squares

$= (x^2 - y^2)(x^2 + y^2)$ … another difference of two squares occurs

$= (x - y)(x + y)(x^2 + y^2)$

(ii) $12x^2 - 75y^2 = 3(4x^2 - 25y^2)$ … find the highest common factor by inspection

$= 3[(2x)^2 - (5y)^2]$ … write as the difference of two squares

$= 3\,(2x - 5y)(2x + 5y)$

Example 2

Simplify $\dfrac{x^2 - 9y^2}{3x + 9y}$

$$\frac{x^2 - 9y^2}{3x + 9y} = \frac{(x - 3y)(x + 3y)}{3(x + 3y)} = \frac{x - 3y}{3}$$

4. Factorising quadratic expressions

We can factorise quadratic expressions of the form $ax^2 + bx + c$ using either

(i) trial and error **or**

(ii) when the coefficients are large or irrational, the quadratic formula.

(i) $x^2 + 3x - 18 = (x \pm ?)(x \pm ?)$ … checking all the factor pairs of -18

$= (x + 6)(x - 3)$ … $(\pm1, \pm18), (\pm2, \pm9), (\pm3, \pm6)$ … $(+6, -3)$ is the factor pair which, when added, produces $+3$ for the middle term.

(ii) $3x^2 - 17x + 20 = ax^2 + bx + c \rightarrow a = 3, b = -17, c = 20$.

Using $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$,

$$x = \frac{17 \pm \sqrt{(-17)^2 - 4.3.20}}{2.3} = \frac{17 \pm \sqrt{289 - 240}}{6}$$

$$\Rightarrow x = \frac{17 \pm 7}{6} = \left(\frac{24}{6} \text{ or } \frac{10}{6}\right) = \left(4 \text{ or } \frac{5}{3}\right)$$

If $x = 4$, then $(x - 4)$ is the factor.

And if $x = \frac{5}{3} \Rightarrow 3x = 5$, therefore the second factor is $(3x - 5)$.

$\therefore \quad 3x^2 - 17x + 20 = (3x - 5)(x - 4)$.

If $ax^2 + bx + c = 0$,

then $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Example 3

Factorise (i) $3x^2 + 10x + 8$ (ii) $x^2 - 2\sqrt{2}x - 6$

(i) $3x^2 + 10x + 8 = (3x \pm ?)(x \pm ?)$ … the factor pairs of 8 are $(\pm1, \pm8), (\pm2, \pm4)$

$= (3x + 4)(x + 2)$ … producing a middle term of $+10x$

(ii) $x^2 - 2\sqrt{2}x - 6 = ax^2 + bx + c \rightarrow a = 1, b = -2\sqrt{2}, c = -6$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \rightarrow x = \frac{2\sqrt{2} \pm \sqrt{(-2\sqrt{2})^2 - 4(1)(-6)}}{2.1} = \frac{2\sqrt{2} \pm \sqrt{32}}{2}$$

Therefore, $x = \dfrac{2\sqrt{2} \pm 4\sqrt{2}}{2} = \sqrt{2} \pm 2\sqrt{2} = 3\sqrt{2}$ or $(-\sqrt{2})$.

The factors are $(x + \sqrt{2})$ and $(x - 3\sqrt{2})$.

5. Factorising expressions of the form $x^3 - y^3$ and $x^3 + y^3$

We can show by long division that $(x - y)$ is a factor of $x^3 - y^3$, creating a second factor $x^2 + xy + y^2$.

$$\begin{array}{r|l}
 & x^2 + xy + y^2 \\
\hline
x - y & x^3 \qquad\qquad\qquad - y^3 \\
 & \underline{x^3 - x^2y} \\
 & \quad + x^2y \qquad\quad - y^3 \\
 & \quad \underline{+ x^2y - xy^2} \\
 & \qquad\quad + xy^2 - y^3 \\
 & \qquad\quad \underline{+ xy^2 - y^3}
\end{array}$$

Therefore, we can write
$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$.

Similarly, we have that
$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$.

If we can write a polynomial in one of these forms, we can use these factor pairs as templates to find its factors.

For example, we can write

(i) $27x^3 + y^3 = (3x)^3 + y^3$

(ii) $64x^3 - 125y^3 = (4x)^3 - (5y)^3$.

$x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

Also

$(ax)^3 - (by)^3 = (ax - by)(a^2x^2 + abxy + b^2y^2)$

$(ax)^3 + (by)^3 = (ax + by)(a^2x^2 - abxy + b^2y^2)$

Example 4

Factorise (i) $a^3 + 8b^3$ (ii) $64c^3 - 125d^3$

(i) $a^3 + 8b^3 = a^3 + (2b)^3$ … note: let $x = a$ and $y = 2b$ in the box on previous page

$= (a + 2b)(a^2 - 2ab + 4b^2)$

(ii) $64c^3 - 125d^3 = (4c)^3 - (5d)^3$ … note: let $x = 4c$ and $y = 5d$

$= (4c - 5d)[(4c)^2 + (4c)(5d) + (5d)^2]$

$= (4c - 5d)(16c^2 + 20cd + 25d^2)$

Exercise 1.3

Using the highest common factor, factorise each of the following:

1. $5x^2 - 10x$ **2.** $6ab - 12bc$ **3.** $3x^2 - 6xy$

4. $2x^2y - 6x^2z$ **5.** $2a^3 - 4a^2 + 8a$ **6.** $5xy^2 - 20x^2y$

7. $2a^2b - 4ab^2 + 12abc$ **8.** $3x^2y - 9xy^2 + 15xyz$ **9.** $4\pi r^2 + 6\pi rh$

Factorise each of the following by grouping terms.

10. $3a(2b - c) - 4(2b - c)$ **11.** $x^2 - ax + 3x - 3a$

12. $2c^2 - 4cd + c - 2d$ **13.** $8ax + 4ay - 6bx - 3by$

14. $7y^2 - 21by + 2ay - 6ab$ **15.** $6xy + 12yz - 8xz - 9y^2$

16. $6x^2 - 3y(3x - 2a) - 4ax$ **17.** $3ax^2 - 3ay^2 - 4bx^2 + 4by^2$

Using the difference of two squares, factorise the following:

18. $a^2 - b^2$ **19.** $x^2 - 4y^2$ **20.** $9x^2 - y^2$

21. $16x^2 - 25y^2$ **22.** $36x^2 - 25$ **23.** $1 - 36x^2$

24. $49a^2 - 4b^2$ **25.** $x^2y^2 - 1$ **26.** $4a^2b^2 - 16c^2$

27. $3x^2 - 27y^2$ **28.** $45 - 5x^2$ **29.** $45a^2 - 20$

30. $(2x + y)^2 - 4$ **31.** $(3a - 2b)^2 - 9$ **32.** $a^4 - b^4$

Factorise each of the following quadratic expressions:

33. $x^2 + 9x + 14$ **34.** $2x^2 + 7x + 3$ **35.** $2x^2 + 11x + 14$

36. $x^2 - 9x + 14$ **37.** $x^2 - 11x + 28$ **38.** $2x^2 - 7x + 3$

39. $3x^2 - 17x + 20$ **40.** $7x^2 - 18x + 8$ **41.** $2x^2 - 7x - 15$

42. $3x^2 + 11x - 20$ **43.** $12x^2 - 11x - 5$ **44.** $6x^2 + x - 15$

45. $3x^2 + 13x - 10$ **46.** $6x^2 - 11x + 3$ **47.** $36x^2 - 7x - 4$

48. $15x^2 - 14x - 8$ **49.** $6y^2 + 11y - 35$ **50.** $12x^2 + 17xy - 5y^2$

51. Using the quadratic formula, factorise each of the following:

(i) $x^2 + 3\sqrt{3}x + 6$ (ii) $x^2 + 2\sqrt{5}x - 15$ (iii) $2x^2 - 5\sqrt{2}x - 6$

52. Using both the sum and the difference of two cubes, factorise the following:

(i) $a^3 + b^3$ (ii) $a^3 - b^3$ (iii) $8x^3 + y^3$

Factorise each of the expressions in numbers (53–55):

53. (i) $27x^3 - y^3$ (ii) $x^3 - 64$ (iii) $8x^3 - 27y^3$

54. (i) $8 + 27k^3$ (ii) $64 - 125a^3$ (iii) $27a^3 + 64b^3$

55. (i) $a^3 - 8b^3c^3$ (ii) $5x^3 + 40y^3$ (iii) $(x + y)^3 - z^3$

Section 1.4 Simplifying algebraic fractions

Algebraic fractions are added, subtracted, multiplied and divided in the same way as numerical fractions.

Revision:

(i) $\frac{2}{5} + \frac{3}{7} = \frac{14}{35} + \frac{15}{35} = \frac{29}{35}$ (fractions can only be added or subtracted when they have the same denominator.)

(ii) $\frac{2}{5} \times \frac{3}{7} = \frac{6}{35}$ (fractions are multiplied by multiplying the numerators and denominators separately.)

(iii) $\frac{2}{5} \div \frac{3}{7} = \frac{2}{5} \times \frac{7}{3} = \frac{14}{15}$ (fractions are divided by changing the division into a product.)

Note: $\frac{{}^{2}\cancel{6} \times 12}{{}_{1}\cancel{3}} = 24$ $\frac{{}^{2}\cancel{6} + \cancel{12}^{4}}{{}_{1}\cancel{3}} = 6$

Similarly with algebraic terms;

(i) $\frac{2}{x+1} - \frac{2x}{2x+3} = \frac{2(2x+3)}{(x+1)(2x+3)} - \frac{2x(x+1)}{(x+1)(2x+3)}$ … getting a common denominator

$= \frac{4x + 6 - 2x^2 - 2x}{(x+1)(2x+3)} = \frac{-2x^2 + 2x + 6}{(x+1)(2x+3)}$ … simplifying the numerator

(ii) $\frac{2}{x+1} \times \frac{2x}{2x+3} = \frac{4x}{(x+1)(2x+3)}$ … multiplying numerators and multiplying denominators

(iii) $\frac{2}{x+1} \div \frac{2x}{2x+3} = \frac{2}{x+1} \times \frac{2x+3}{2x} = \frac{2(2x+3)}{2x(x+1)} = \frac{(2x+3)}{x(x+1)}$ … changing division to multiplication and then dividing above and below by a common factor

So, in general when dealing with algebraic fractions;

(i) A common denominator is needed to add or subtract fractions.
(ii) A fraction can be reduced (simplified) only if the numerator and denominator have a common factor.
(iii) If the denominator or numerator contain fractions added or subtracted, they must be reduced into a single fraction first before proceeding.
(iv) To divide fractions, we multiply by the denominator inverted.

Example 1

Simplify (i) $\frac{5ax}{15a + 10a^2}$ (ii) $\frac{t^2 + 3t - 4}{t^2 - 16}$ (iii) $\frac{\frac{5}{8} + y}{\frac{1}{8}}$

(i) $\frac{5ax}{15a + 10a^2} = \frac{(\cancel{5a})x}{(\cancel{5a})(3 + 2a)} = \frac{x}{3 + 2a}$

(ii) $\frac{t^2 + 3t - 4}{t^2 - 16} = \frac{(\cancel{t + 4})(t - 1)}{(t - 4)(\cancel{t + 4})} = \frac{t - 1}{t - 4}$

(iii) $\frac{\frac{5}{8} + y}{\frac{1}{8}} = \left(\frac{5}{8} + y\right).8 = 5 + 8y$

Example 2

Simplify each of the following

(i) $\frac{6y}{x(x + 4y)} - \frac{3}{2x}$ (ii) $\frac{5}{3x - 4} + \frac{2x + 5}{3}$

(i) $\frac{6y}{x(x + 4y)} - \frac{3}{2x} = \frac{2(6y)}{2x(x + 4y)} - \frac{3(x+4y)}{2x(x + 4y)}$

$= \frac{2(6y) - 3(x+4y)}{2x(x + 4y)} = \frac{12y - 3x - 12y}{2x(x + 4y)}$

$= \frac{-3x}{2x(x + 4y)} = \frac{-3}{2(x + 4y)}$

(i) $\dfrac{5}{3x-4} + \dfrac{2x+5}{3} = \dfrac{3(5)}{3(3x-4)} + \dfrac{(2x+5)(3x-4)}{3(3x-4)}$ … common denom.: $3(3x-4)$

$$= \frac{15 + 6x^2 + 7x - 20}{3(3x-4)}$$

$$= \frac{6x^2 + 7x - 5}{3(3x-4)} = \frac{(2x-1)(3x+5)}{3(3x-4)}$$

Example 3

Simplify $\dfrac{y - \dfrac{x^2+y^2}{y}}{\dfrac{1}{x} - \dfrac{1}{y}}$.

$$\frac{y - \dfrac{x^2+y^2}{y}}{\dfrac{1}{x} - \dfrac{1}{y}} = \frac{\dfrac{y^2 - (x^2+y^2)}{y}}{\dfrac{(y-x)}{xy}} = \frac{\dfrac{-x^2}{y}}{\dfrac{(y-x)}{xy}} = \frac{-x^2}{y} \times \frac{xy}{(y-x)}$$

$$= \frac{-x^3y}{y(y-x)} = \frac{-x^3}{y-x}$$

Exercise 1.4

1. Simplify each of the following fractions:

(i) $\dfrac{8y}{2y^3}$ (ii) $\dfrac{7a^6b^3}{14a^5b^4}$ (iii) $\dfrac{(2x)^2}{4x}$ (iv) $\dfrac{7y+2y^2}{7y}$ (v) $\dfrac{5ax}{15a+10a^2}$

2. Express each of the following as a single fraction:

(a) $\dfrac{2x}{5} + \dfrac{4x}{3}$ (b) $\dfrac{3x}{5} - \dfrac{x}{2}$ (c) $\dfrac{2x+3}{4} + \dfrac{x}{3}$

(d) $\dfrac{x+1}{4} + \dfrac{2x-1}{5}$ (e) $\dfrac{3x-4}{6} - \dfrac{2x+1}{3}$ (f) $\dfrac{3x-2}{6} - \dfrac{x-3}{4}$

(g) $\dfrac{5x-1}{4} - \dfrac{2x-4}{5}$ (h) $\dfrac{3x+5}{6} - \dfrac{2x+3}{4} - \dfrac{1}{12}$ (i) $\dfrac{3x-2}{4} + \dfrac{3}{5} - \dfrac{2x-1}{10}$

(j) $\dfrac{1}{3x} + \dfrac{1}{5x}$ (k) $\dfrac{3}{4x} - \dfrac{5}{8x}$ (l) $\dfrac{1}{x} + \dfrac{1}{x+3}$

(m) $\dfrac{2}{x+2} + \dfrac{3}{x+4}$ (n) $\dfrac{2}{x-2} + \dfrac{3}{2x-1}$ (o) $\dfrac{5}{3x-1} - \dfrac{2}{x+3}$

(p) $\dfrac{3}{2x-7} - \dfrac{1}{5x+2}$ (q) $\dfrac{2}{3x-5} - \dfrac{1}{4}$ (r) $\dfrac{5}{2x-1} - \dfrac{3}{x-2}$

(s) $\dfrac{x}{x-y} - \dfrac{y}{x+y}$ (t) $\dfrac{3}{x} + \dfrac{4}{3y} - \dfrac{2}{3xy}$ (u) $\dfrac{3}{x} - \dfrac{2}{x-1} - \dfrac{4}{x(x-1)}$

$$(x^2 - a^2) = (x - a)(x + a)$$

3. By factorising the numerator and the denominator fully, simplify each of the following.

(i) $\dfrac{2z^2 - 4z}{2z^2 - 10z}$ (ii) $\dfrac{y^2 + 7y + 10}{y^2 - 25}$ (iii) $\dfrac{t^2 + 3t - 4}{t^2 - 3t + 2}$

(iv) $\dfrac{x}{x^2 - 4} - \dfrac{1}{x+2}$ (v) $\dfrac{2}{a+3} - \dfrac{a+2}{a^2 - 9}$ (vi) $\dfrac{x-1}{x^2 - 4} + \dfrac{1}{x-2}$

4. By factorising the denominator, simplify each of the following:

(i) $\dfrac{10}{2x^2 - 3x - 2} - \dfrac{2}{x-2}$ (ii) $\dfrac{x+2}{2x^2 - x - 1} - \dfrac{1}{x-1}$

5. Simplify the following:

(i) $\dfrac{1}{x^2 - 9} - \dfrac{2}{x^2 - x - 6}$ (ii) $\dfrac{3}{x^2 + x - 2} - \dfrac{2}{x^2 + 3x + 2}$

(iii) $\dfrac{2}{6x^2 - 5x - 4} - \dfrac{3}{9x^2 - 16}$ (iv) $\dfrac{1}{xy - x^2} - \dfrac{1}{y^2 - xy}$

6. Simplify each of the following complex fractions:

(i) $\dfrac{\frac{1}{2} + \frac{3}{4}}{\frac{1}{4}}$ (ii) $\dfrac{\frac{2}{3} + \frac{5}{6}}{\frac{3}{8}}$ (iii) $\dfrac{x - \frac{1}{x}}{1 + \frac{1}{x}}$

7. Simplify each of these:

(i) $\dfrac{\frac{1}{x} + 1}{\frac{1}{x} - 1}$ (ii) $\dfrac{\frac{1}{x^2} - 4}{\frac{1}{x} - 2}$ (iii) $\dfrac{x + y}{\frac{1}{x} + \frac{1}{y}}$

8. By expressing the numerator as a single fraction, simplify the following fractions:

(i) $\dfrac{4y - \frac{3}{2}}{2}$ (ii) $\dfrac{2 - \frac{1}{x}}{2}$ (iii) $\dfrac{3x + \frac{1}{x}}{2}$ (iv) $\dfrac{y + \frac{1}{4}}{\frac{1}{2}}$

9. By expressing the numerator and the denominator as single fractions, write the following fractions in their simplest forms.

(i) $\dfrac{z - \frac{1}{3}}{z - \frac{1}{2}}$ (ii) $\dfrac{2x + \frac{1}{2}}{x + \frac{1}{4}}$ (iii) $\dfrac{z - \frac{1}{2z}}{z - \frac{1}{3z}}$ (iv) $\dfrac{x - \frac{1}{x+1}}{x - 1}$

10. Simplify each of the following.

(i) $\dfrac{1 + \frac{2}{x}}{\frac{x+2}{x-2}}$ (ii) $\dfrac{2 + \frac{1}{x}}{2x^2 + x}$ (iii) $\dfrac{x + \frac{2x}{x-2}}{1 + \frac{4}{(x+2)(x-2)}}$

11. Simplify each of the following.

(i) $\dfrac{\frac{a+b}{a-b} - \frac{a-b}{a+b}}{1 + \frac{a-b}{a+b}}$ (ii) $\dfrac{x + \frac{3}{x}}{x - \frac{9}{x^3}}$ (iii) $\dfrac{9 - \frac{1}{y^2}}{9 + \frac{6}{y} + \frac{1}{y^2}}$

12. Show that $\dfrac{3x - 5}{x - 2} + \dfrac{1}{2 - x}$ simplifies to a constant when $x \neq 2$.

Section 1.5 Algebraic identities

The word identity occurs in many different areas of mathematics. It is used in trigonometry, in sets, in functions and in algebra.

> In an **identity,** all coefficients of like powers are equal.
>
> An identity must be true **for all values** of the independent variable.

If $3x + 7 = ax + b$ **for all values of x**, this is called an **algebraic identity**.

We can conclude that for this to be true, then $a = 3$ and $b = 7$.

When two expressions are equal **for all values of x**, then the resulting equation is an identity. All coefficients of like powers of x in an identity are equal.

Generally, if $ax^3 + bx^2 + cx + d = px^3 + qx^2 + rx + s$ for all values of x,

then $a = p, b = q, c = r, d = s$.

Also, if $ax^3 + bx^2 + cx + d = qx^2 + s$ for all values of x,

then $a = c = 0$ and $b = q, d = s$.

This property is used to find unknown coefficients in certain equations.

Example 1

Find the values of a and b given that $(2x + a)^2 = 4x^2 + 12x + b$, **for all values of x**.

Given $(2x + a)^2 = 4x^2 + 12x + b$ for all values of x,

$4x^2 + 4ax + a^2 = 4x^2 + 12x + b$

$4a = 12$ (comparing like powers of x) $\therefore\ a = 3$

and $a^2 = b$ (comparing the constant terms) $\therefore\ 3^2 = 9 = b$.

Example 2

If $3t^2x - 3px + c - 2t^3 = 0$ **for all values of x**, find c in terms of p.

Given $3t^2x - 3px + c - 2t^3 = 0$ for all values of x,

$\therefore\ (3t^2 - 3p)x + c - 2t^3 = (0)x + (0)$ … writing both sides as polynomials in x

$\therefore\ 3t^2 - 3p = 0$ (comparing like powers of x) $\therefore\ t = \sqrt{p}$

and $c - 2t^3 = 0$ (comparing the constant terms) $\therefore\ c = 2t^3$

$$\therefore\ c = 2(\sqrt{p})^3 = 2p^{\frac{3}{2}}$$

Algebraic identities can also be used to create **partial fractions** from a given fraction.

For example, $\dfrac{1}{(x+1)(x-2)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x-2)}$ where A and $B \in Q$.

Example 3

Given $\dfrac{1}{(x+1)(x-2)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x-2)}$ for all values of x, find the values of A and B.

$$\frac{1}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)} = \frac{A(x-2) + B(x+1)}{(x+1)(x-2)}$$

$\therefore\ 1 = A(x - 2) + B(x + 1)$ for all values of x.

We can find values of A and B using two different methods.

Method 1: Since this equation must be true for all values of x, by picking two suitable values of x, A and B can be easily evaluated.

Since $1 = A(x - 2) + B(x + 1)$

Let $x = 2$ $\therefore\ 1 = A(0) + B(2 + 1)$ $\therefore\ B = \frac{1}{3}$

Let $x = -1$ $\therefore\ 1 = A(-1 - 2) + B(0)$ $\therefore\ 1 = -3A \Rightarrow A = -\frac{1}{3}$

Method 2: Equating coefficients of like powers and then solve using simultaneous equations.

$$1 = A(x-2) + B(x+1)$$
$$1 = Ax - 2A + Bx + B$$
$$1 = Ax + Bx - 2A + B$$
$$1 = x(A+B) - 2A + B$$
$$x(0) + 1 = x(A+B) - 2A + B \text{ for all values of } x.$$

$\therefore \quad 0 = A + B$ and $1 = -2A + B$ … equating like powers of x and constant terms

$\therefore \quad A = -B$ and using substitution, $1 = -2(-B) + B = 3B$.

$\therefore \quad B = \frac{1}{3}$ and $A = -\frac{1}{3}$

$$\therefore \quad \frac{1}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)} = \frac{-1}{3(x+1)} + \frac{1}{3(x-2)}$$

Algebraic identities and factors

If $x^2 - ax + b$ is a factor of $x^3 + 2ax^2 + 4bx + c$,
using **algebraic identities** we can find a relationship between the coefficients a, b and c.

It is important to realise that when a factor is divided into an expression, there can be no remainder by definition of a factor.

$$\begin{array}{r|l} & x + 3a \\ \hline x^2 - ax + b & x^3 + 2ax^2 + 4bx + c \\ & \underline{x^3 - ax^2 + bx} \\ & \quad 3ax^2 + 3bx + c \\ & \quad \underline{3ax^2 - 3a^2x + 3ab} \\ \end{array}$$

$$\text{Remainder} = x(3b + 3a^2) + c - 3ab$$

Since there can be no remainder, we can conclude that **for all values of x,**

(i) $3b + 3a^2 = 0$, i.e. $b = -a^2$ and

(ii) $c - 3ab = 0$, i.e. $c = 3ab$

The same results can be obtained by letting the missing factor $= (x + k)$.

$$\therefore \quad (x+k)(x^2 - ax + b) = x^3 + 2ax^2 + 4bx + c$$

Expanding the left-hand side we get,

$$x^3 - ax^2 + bx + kx^2 - akx + bk = x^3 + 2ax^2 + 4bx + c \quad \text{… \textbf{for all values of } } \boldsymbol{x}.$$

$$\therefore \quad x^3 + (k-a)x^2 + (b-ak)x + bk = x^3 + 2ax^2 + 4bx + c \quad \text{… grouping like terms.}$$

Comparing coefficients of x^2: $(k - a) = 2a$, hence $k = 3a$ as above.

Comparing coefficients of x: $(b - ak) = 4b$
$(b - 3a^2) = 4b$, hence $3b = -3a^2$, i.e. $b = -a^2$ as above.

Finally, comparing constant terms: $bk = c$
$\therefore \quad 3ab = c$, again the same as above.

Example 4

Given that $(x - t)^2$ is a factor of $x^3 + 3px + c$, show that $p = -t^2$ and $c = 2t^3$.

$(x - t)^2 = x^2 - 2xt + t^2$ and using long division we get;

$$\begin{array}{r l}
 & \quad x + 2t \\
x^2 - 2xt + t^2 & \overline{)\,x^3 \qquad\quad + 3px + c} \qquad \text{(note: there is no } x^2 \text{ term)} \\
 & \underline{x^3 - 2tx^2 + t^2x} \\
 & \qquad 2tx^2 - t^2x + 3px + c \\
 & \qquad 2tx^2 + (3p - t^2)x + c \\
 & \qquad \underline{2tx^2 - 4t^2x + 2t^3} \\
 & \qquad (3p + 3t^2)x + c - 2t^3 \qquad (= \text{remainder})
\end{array}$$

Since we should get no remainder, $(3p + 3t^2)x + c - 2t^2 = 0$ **for all values of x**.

$\therefore \quad (3p + 3t^2)x + c - 2t^3 = (0)x + 0 \qquad$ for all values of x.

$\therefore \quad 3p + 3t^2 = 0 \rightarrow p = -t^2$

and $\quad +c - 2t^3 = 0 \rightarrow c = 2t^3$.

(Note: The factors of $x^3 + 3px + c$ are $x^2 - 2xt + t^2$ and $x + 2t$.)

Example 5

$2x - \sqrt{3}$ is a factor of $4x^2 - 2(1 + \sqrt{3})x + \sqrt{3}$; find the second factor.

Let the second factor take the form $(ax + b)$.
(Note: a needs to be introduced because the coefficient of x^2 is 4; it should be clear that $a = 2$)

Then $(2x - \sqrt{3})(ax + b) = 4x^2 - 2(1 + \sqrt{3})x + \sqrt{3}$

$2ax^2 + 2bx - \sqrt{3}ax - \sqrt{3}b = 4x^2 - 2(1 + \sqrt{3})x + \sqrt{3}$

$2ax^2 + (2b - \sqrt{3}a)x - \sqrt{3}b = 4x^2 - 2(1 + \sqrt{3})x + \sqrt{3}$

Equating coefficients of like powers,

(i) $(x^2) \ldots 2a = 4 \Rightarrow a = 2$

(ii) $(x) \ldots + (2b - \sqrt{3}a) = -2(1 + \sqrt{3})$
since $a = 2, \Rightarrow 2b - 2\sqrt{3} = -2 - 2\sqrt{3}$
$\therefore \quad b = -1$

(iii) (comparing the constants) $\ldots -\sqrt{3}b = \sqrt{3}$, verifying that $b = -1$.
Therefore the second factor is $2x - 1$.

Exercise 1.5

1. If $ax^2 + bx + c = (2x - 3)(3x + 4)$ for all values of x, find the values of a, b and c.

2. If $(3x - 2)(x + 5) = 3x^2 + px + q$ for all values of x, find the values of p and q.

3. If $x^2 + 6x + 16 = (x + a)^2 + b$ for all values of x, find the values of a and b.

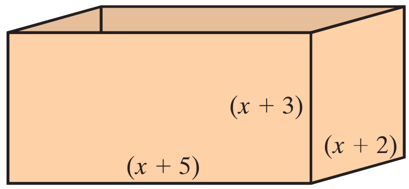
4. Find the real numbers a and b such that $x^2 + 4x - 6 = (x + a)^2 + b$ for all $x \in R$.

5. If $2x^2 + 5x + 6 = p(x + q)^2 + r$ for all values of x, find the values of p, q and r.

6. Find the values of a and b if $(2x + a)^2 = 4x^2 + 12x + b$, for all x.

7. If $x^2 - 4x - 5 = (x - n)^2 - m$ for all x, find the values of m and n.

8. The volume of this closed box, V, is given by the function $V(x) = ax^3 + bx^2 + cx + d$, where a, b, c and $d \in Z$.



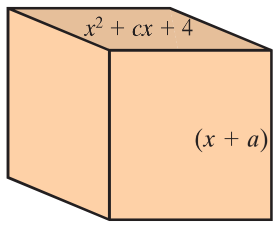
   (i) Find the values for a, b, c and d.

   The external surface area, $S(x)$, is given by the equation $S(x) = px^2 + qx + r$, where p, q and $r \in Z$.

   (ii) Find the values of p, q and r.

9. If $3(x - p)^2 + q = 3x^2 - 12x + 7$ for all x, find the values of p and q.

10. The volume of a solid box is given by $V(x) = x^3 + 12x^2 + bx + 30$.



    If the top of the box has an area of $x^2 + cx + 4$ and the height is $x + a$, find the values of a, b and c.

11. If $(x - 4)^3 = x^3 + px^2 + qx - 64$ for all x, find the values of the constants p and q.

12. If $(x + a)(x^2 + bx + 2) = x^3 - 2x^2 - x - 6$ for all x, find the values of the constants a and b.

13. Find the values of b and c given that $(x - 2)(x^2 + bx + c) = x^3 + 2x^2 - 5x - 6$ for all values of x.

14. Given that $(5a - b)x + b + 2c = 0$ for all values of x, find a in terms of c.

15. If $(4x + r)(x^2 + s) = 4x^3 + px^2 + qx + 2$ for all x, find a value for pq.

16. $(x + s)(x - s)(ax + t) = ax^3 + bx^2 + cx + d$ for all values of x.
Using this identify, show that $bc = ad$.

17. If $\dfrac{1}{(x + 1)(x - 1)} = \dfrac{A}{(x + 1)} + \dfrac{B}{(x - 1)}$ for all x, find values for A and B.

18. If $\dfrac{1}{(x + 2)(x - 3)} = \dfrac{C}{(x + 2)} + \dfrac{D}{(x - 3)}$ for all x, find values for C and D.

19. Write $\dfrac{1}{(x + 1)(x + 4)}$ as the partial fractions $\dfrac{A}{(x + 1)} + \dfrac{B}{(x + 4)}$.

20. If $(x - 3)^2$ is a factor of $x^3 + ax + b$, find the value of a and the value of b.

21. If $(x - 2)^2$ is a factor of $x^3 + px + q$, find the value of p and the value of q.

22. Given that $(x^2 - 4)$ is a factor of $x^3 + cx^2 + dx - 12$, find the values of the coefficients c and d.
Hence factorise the cubic polynomial fully.

23. If $(x^2 + b)$ is a factor of $x^3 - 3x^2 + bx - 15$, find the value of b.

24. If $x^2 - px + 9$ is a factor of $x^3 + ax + b$, express (i) a (ii) b in terms of p.
Hence find the values of p for which $a + b = 17$.

25. If $x^2 - kx + 1$ is a factor of $ax^3 + bx + c$, show that $c^2 = a(a - b)$.

26. If $(x - a)^2$ is a factor of $x^3 + 3px + c$, show that (i) $p = -a^2$ (ii) $c = 2a^3$.

27. If $x^2 + ax + b$ is a factor of $x^3 - k$, show that (i) $a^3 = k$ (ii) $b^3 = k^2$.

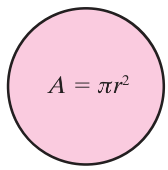
28. Show by long division that $2x - \sqrt{3}$ is a factor of $4x^2 - 2(1 + \sqrt{3})x + \sqrt{3}$ and hence find the second factor.

29. Find the values of A, B and C such that
$5x + 3 = Ax(x + 3) + Bx(x - 1) + C(x - 1)(x + 3)$ for all values of x.

Section 1.6 Manipulating formulae

The area of a disc is given by the formula $A = \pi r^2$.

A is said to be defined **in terms of** r (r is the independent variable).



If we know r, we can find A.
For example, if $r = 3$, then $A = 9\pi$; if $r = 8$, then $A = 64\pi$, etc..

In some cases we may need to find r in terms A, i.e. make r the **subject** of the equation.

$$\therefore \quad \pi r^2 = A \Rightarrow r^2 = \frac{A}{\pi} \Rightarrow r = \sqrt{\frac{A}{\pi}} \quad \ldots \; r \text{ is now defined in terms of } A.$$

Thus, if we know the area of the circle, we can find the radius.

For example, if $A = 9\pi$, then $r = \sqrt{\frac{A}{\pi}} \Rightarrow r = \sqrt{\frac{9\pi}{\pi}} = \sqrt{9} = 3$.

Example 1

(i) If $v^2 = u^2 + 2as$, express a in terms of v, u and s.

(ii) If $\sqrt{\frac{x+y}{x-y}} = \frac{1}{2}$, express y in terms of x. Hence find the value of y when $x = 5$.

(i)
$$\begin{aligned} v^2 &= u^2 + 2as \\ u^2 + 2as &= v^2 \\ 2as &= v^2 - u^2 \\ \therefore\ a &= \frac{v^2 - u^2}{2s} \end{aligned}$$

(ii) $\sqrt{\frac{x+y}{x-y}} = \frac{1}{2}$

$\frac{x+y}{x-y} = \frac{1}{4}$ … squaring both sides

$4(x+y) = x - y$ … multiplying both sides by $4(x-y)$

$4x + 4y = x - y$ … expanding the left-hand side

$y + 4y = x - 4x$ … gathering only y terms on the left-hand side

$5y = -3x$

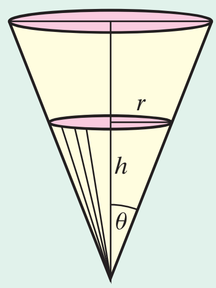
$y = \frac{-3x}{5}$ … finding y in terms of x

When $x = 5$, $y = \frac{-3(5)}{5} = -3$.

Example 2

A container in the shape of an inverted cone is used to hold liquid.

Given $\tan\theta = \frac{r}{h}$, express the volume, V, in terms of the depth, h, of the liquid and the angle θ.



Volume of a cone, $V = \frac{1}{3}\pi r^2 h$

But $\tan\theta = \frac{r}{h} \Rightarrow h\tan\theta = r$

$$V = \tfrac{1}{3}\pi r^2 h = \frac{\pi}{3}(h\tan\theta)^2 h = \frac{\pi}{3}h^3\tan^2\theta$$

Example 3

Given $x = \frac{t+4}{3t+1}$, find t in terms of x.

$$x = \frac{t+4}{3t+1}$$

$x(3t + 1) = t + 4$ ... multiplying both sides by $3t + 1$

$3tx + x = t + 4$ ... expanding the left-hand side

$3tx - t = 4 - x$ ... gathering only t terms on the left-hand side

$t(3x - 1) = 4 - x$ ... factorising

$t = \frac{4-x}{3x-1}$ ... dividing both sides by $3x - 1$

Exercise 1.6

1. In each of the following, express x in terms of the other variables.

(i) $3x - 2y = 4$ (ii) $2x - b = 4c$ (iii) $5x - 4 = \frac{y}{2}$

(iv) $5(x - 3) = 2y$ (v) $3y = \frac{x}{3} - 2$ (vi) $xy = xz + yz$

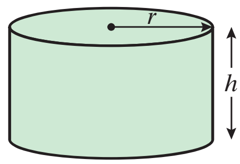
2. Express x in terms of the other variables in each of the following:

(i) $2x - \frac{y}{3} = \frac{1}{3}$ (ii) $z = \frac{y - 2x}{3}$ (iii) $\frac{a}{x} - b = c$

3. (a) The volume of a cylinder is given by $V = \pi r^2 h$.
Find the radius r in terms of V and h.

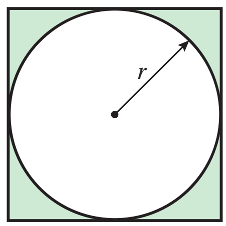
(b) The curved surface area of a cylinder is given by $A = 2\pi rh$.
Find the radius r in terms of A and h.

(c) Hence show that $A^2 = 4\pi hV$.



4. A circle of radius r is drawn inside a square as shown.

(a) Find the area of the circle, A.
(b) Find the area of the square in terms of r.
(c) Hence find an expression for the area of the shaded corners in terms of r.
(d) If the side of the square is doubled while the radius of the circle is halved, find an expression for the shaded area in terms of r.
(e) If a circle is circumscribed around the original square, prove that the area of the outer disc is twice the area of the inner disc.

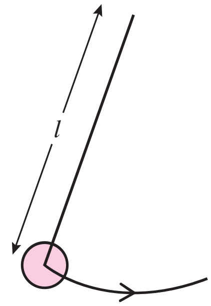


5. A speed camera measures the change in frequency of waves from f to f', caused by a moving car, using the formula $f' = \dfrac{fc}{c-u}$, where c is the speed of the waves and u the speed of the car. Find

(i) the speed of the car, u, in terms of the other variables f', f and c.

(ii) the speed of the waves, c, in terms of the other variables f', f and u.

6. The time taken for one complete cycle of a pendulum is given by $T = 2\pi\sqrt{\dfrac{l}{g}}$, where l is the length of the pendulum and g the acceleration due to gravity.



(i) Find l in terms of the other variables.

(ii) Given that $T = 3\,\text{s}$ and $g = 10\,\text{m s}^{-2}$, calculate the length of the pendulum correct to one decimal place.

7. In each of the following, express a in terms of the other variables:

(i) $\dfrac{x}{y} = \dfrac{a+b}{a-b}$ (ii) $bc - ac = ac.$

8. Express v in terms of the other variables in each of the following:

(i) $y = \dfrac{3(u-v)}{4}$ (ii) $S = \dfrac{t}{2}(u+v)$

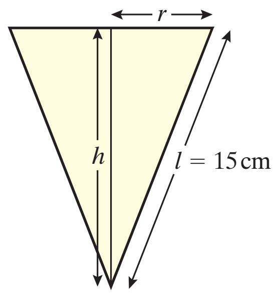
9. The future value of €P, invested for 3 years at i%, is given by the formula $A = P\left(1 + \dfrac{i}{100}\right)^3$. Find i in terms of P and A.

If €2500 invested 3 years ago has a present value of €2650, find the rate of interest, i (correct to one place of decimals).

10. Write c in terms of the other variables in each of the following.

(i) $d = \sqrt{\dfrac{a-b}{ac}}$ (ii) $b = \dfrac{2c-1}{c-1}$

11. A cone has a radius r cm and a vertical height h cm. If the slant height $l = 15$ cm, and using Pythagoras' theorem:



(i) Express h in terms of r.

(ii) Hence find the value of h when $r = 5$ cm.

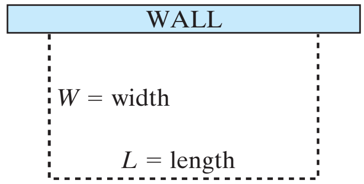
(iii) At what value of h will the radius r be equal to half the slant height l? Give your answer correct to the nearest cm.

12. A farmer has 300 metres of fencing and wants to make a rectangular paddock against an existing wall, as shown, using all of this fencing.

(i) Find the length (L) of the paddock in terms of the width (W).

(ii) Hence find the area of the paddock (A) in terms of the width only.

(iii) Find the dimensions of the paddock if the maximum allowable area is 10,000 m^2.



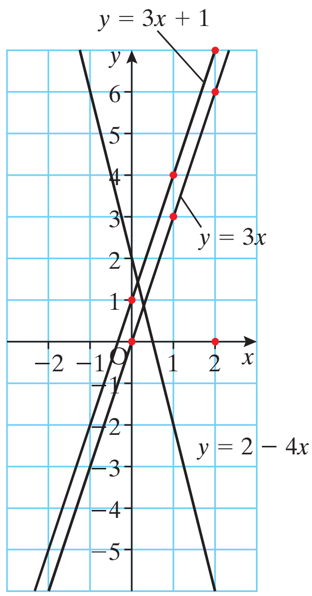
Section 1.7 Algebraic patterns, an introduction

1. Linear Each polynomial function creates a pattern which can be studied both numerically and graphically.

Patterns such as 0, 3, 6, 9, … can be described by the function
$f(x) = (y) = 3x$, where $x \geqslant 0$.

Similarly, 1, 4, 7, 10, … can be described by the function
$f(x) = (y) = 3x + 1$, where $x \geqslant 0$.

Also, 2, −2, −6, −10 … can be represented by the function
$f(x) = (y) = 2 - 4x$, where $x \geqslant 0$.



Each of these number patterns has a constant amount added or subtracted between terms and each, when graphed, produces a straight line as shown.

Functions of the form $f(x) = y = mx + c$ are called **linear** functions.

In a linear pattern, the difference between consecutive terms (called the 1st difference) gives $\boldsymbol{m}$, the slope of the line.

						$\boldsymbol{y = mx + c}$
Pattern	0	3	6	9	12	$f(x) = (y) = 3x$
1st Difference		3	3	3	3	
Pattern	1	4	7	10	13	$f(x) = (y) = 3x + 1$
1st Difference		3	3	3	3	
Pattern	2	−2	−6	−10	−14	$f(x) = (y) = 2 - 4x$
1st Difference		−4	−4	−4	−4	

The starting point of each pattern determines c, the constant term. We note that when $x = 0$, geometrically, we get the y-axis intercept, c.

2. Quadratic Patterns such as $0, 1, 4, 9, 16 \ldots$ do not have a constant amount added or subtracted between terms.
Studying the 1st differences between terms, we get $1, 3, 5, 7, \ldots$.

The **2nd difference** $(3 - 1), (5 - 3), \ldots$ however is a constant, 2.

Such patterns can be represented by the function $f(x) = (y) = x^2$ for $x \geqslant 0$.

Functions of the form $\boldsymbol{f(x) = x^2 + b}$ create similar patterns.

Pattern	0	1	4	9	16	25	$f(x) = (y) = x^2$
1st Difference		1	3	5	7	9	
2nd Difference			2	2	2	2	
Pattern	2	3	6	11	18	27	$f(x) = (y) = x^2 + 2$
1st Difference		1	3	5	7	9	
2nd Difference			2	2	2	2	
Pattern	-3	-2	1	6	13	22	$f(x) = (y) = x^2 - 3$
1st Difference		1	3	5	7	9	
2nd Difference			2	2	2	2	
Pattern	2	1	-2	-7	-14	-23	$f(x) = (y) = 2 - x^2$
1st Difference		-1	-3	-5	-7	-9	
2nd Difference			-2	-2	-2	-2	

As can be seen from the graphs, the curves are symmetrical about the line $x = 0$ (y-axis).

Patterns of the form $f(x) = x^2 + b$ are ∪-shaped.

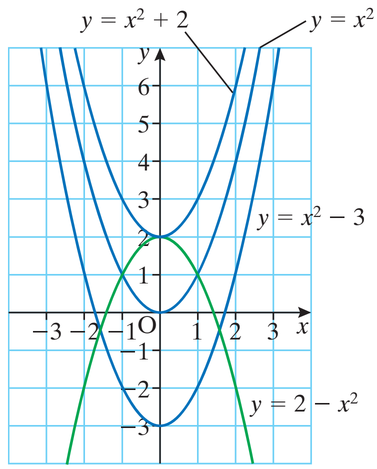
Patterns of the form $f(x) = b - x^2$ are ∩-shaped.

Consider the following pattern.

Pattern	4	7	16	31	52	79
1st Difference		3	9	15	21	27
2nd Difference			6	6	6	6

Because the 2nd difference is a constant, we know this is a quadratic pattern of the form

$\boldsymbol{ax^2 + b}, x \geqslant 0$.



The starting value is 4, i.e. when $x = 0, b = 4$.

> If $f(x) = ax^2 + bx + c$,
> the second difference $= 2a$.

When $x = 1, ax^2 + b = a(1) + 4 = 7 \Rightarrow a = 3$.
(We also note that the second difference is 6, i.e. $2 \times 3 = 2a$)
Thus this pattern of numbers can be represented by the polynomial function $f(x) = 3x^2 + 4$.

Example 1

Examine each of the following patterns of numbers and determine if there is a linear or quadratic relationship between the terms.

Write an algebraic expression for each set of numbers:

(a) $-2, 1, 4, 7, \ldots$ (b) $3, 5, 11, 21, \ldots$

Pattern	-2	1	4	7
1st Difference		3	3	3

(a) Since the first difference is a constant, this indicates a linear relationship $f(x) = ax + b$.
$a = 3$ and $b = -2$,
$\therefore \quad f(x) = 3x - 2, x \geqslant 0$.

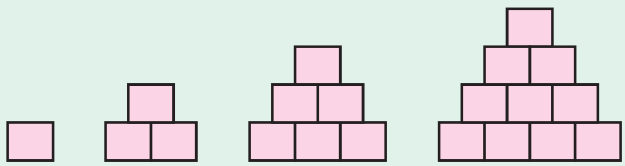
Pattern	3	5	11	21
1st Difference		2	6	10
2nd Difference			4	4

(b) Since the second difference is a constant, this indicates a quadratic relationship
$f(x) = ax^2 + b$.
$4 = 2a \Rightarrow a = 2$ and $b = 3$,
$\therefore \quad f(x) = 2x^2 + 3, x \geqslant 0$.

More complex quadratic patterns of the form $ax^2 + bx + c$ can be formed in two stages; first to identify the quadratic element x^2, then subtracting this from the pattern to form the linear element $bx + c$.

Example 2

Single matchsticks were used to form a sequence of patterns as shown. Find an algebraic quadratic expression for the number of matchsticks needed for each pattern.



How many matchsticks are needed for the 10th pattern?

Counting the matchsticks needed for the first four patterns, we get the number pattern $4, 10, 18, 28, \ldots$

Pattern	4	10	18	28
1st Difference		6	8	10
2nd Difference			2	2

The 2nd difference produces $2, 2, 2, \ldots$ etc.

So there is a quadratic element (x^2) to this pattern.

Trying $f(x) = x^2 + 4$ for $x \geqslant 0$, we get the pattern 4, 5, 8, 13, which is not the required pattern.

Subtracting $f(x) = x^2 = 0, 1, 4, 9, \ldots$ for $x \geqslant 0$ from $4, 10, 18, 28, \ldots$ produces the new pattern $4, 9, 14, 19, \ldots$

Pattern	4	10	18	28
x^2	0	1	4	9
New Pattern	4	9	14	19
1st Difference		5	5	5

The first difference in this sequence is 5.
– a linear relation of the form $5x + 4$ for $x \geqslant 0$.

Combining the two relationships, we get $f(x) = x^2 + 5x + 4$ for $x \geqslant 0$.

To get the 10th pattern, we let $x = 9$ (since our equation is true for $x \geqslant 0$, i.e. it starts at 0). $\therefore$ $f(9) = 9^2 + 5(9) + 4 = 130$ matchsticks are needed.

[Note: If $x \geqslant 1$, we would subtract 1, 4, 9, etc. creating a new sequence $3, 6, 9, \ldots$ with a linear relationship $3x$. Combining, we get $f(x) = x^2 + 3x$ for $x \geqslant 1$, and using this formula, $f(10) = 10^2 + 3(10) = 130$ again.]

Note: Linear and quadratic patterns are studied in greater depth in chapter 4.

Exercise 1.7

1. Examine each of the following patterns of numbers and determine if the pattern has a linear or quadratic relationship.

(a)	$4, 7, 10, 13, 16, \ldots$	(b)	$-2, 2, 6, 10, 14, \ldots$
(c)	$-4, -3, 0, 5, 12, \ldots$	(d)	$2, 1, -2, -7, -14, -23, \ldots$
(e)	$2, 7, 22, 47, \ldots$	(f)	$3, 1, -5, -15, -29, \ldots$
(g)	$1, -4, -19, -44, -79, \ldots$	(h)	$3, -2, -7, -12, -17, \ldots$
(i)	$0, 3, 12, 27, 48, \ldots$	(j)	$5, 17, 37, 65, 101, \ldots$

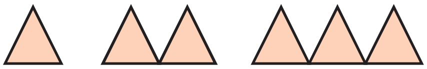
2. Write an algebraic expression to represent each of the followings number patterns.
   (a) $-1, 3, 15, 35, 63, \ldots$ (b) $4, 3, 0, -5, -12, -21, -32, \ldots$

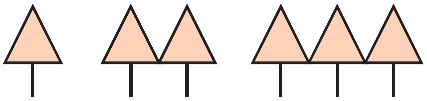
3. Each of the following number patterns can be written in the form $f(x) = ax + b, x \geqslant 0$. Find the values of a and b.
   (i) $2, 7, 12, 17, 22, \ldots$ (ii) $-6, -2, 2, 6, 10, \ldots$
   (iii) $3, 2, 1, 0, -1, -2, \ldots$ (iv) $-2, -7, -12, -17, -22, -27, \ldots$
   (v) $3, 3.5, 4, 4.5, 5, \ldots$ (vi) $-1, -0.8, -0.6, -0.4, -0.2, \ldots$

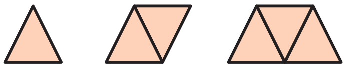
4. If $x \geqslant 3$, find an algebraic linear expression for the pattern $11, 13, 15, 17, 19, \ldots$

5. If $x \geqslant -2$, find a and b such that $f(x) = ax + b$ represents the number pattern $1, 3, 5, 7, 9, \ldots$

6. Convert each of the following designs to number patterns. By finding an algebraic linear expression for the number of matchsticks needed for each design, find the number of matchsticks needed to make the 15th element of each design.

(a) 

(b) 

(c) 

7. A company offers two different billing plans for the purchase of a TV over a number of months. Plan A where the repayments are €35.00 per month with a down-payment of €70.00, or Plan B with repayments of €24.00 per month with a down-payment of €125.00. If x represents the number of months of the plan, write an expression for each billing plan. Write a number sequence representing the cost per month of each plan (A and B) for the first four months. After how many months would both plans have repaid the same amount?

8. A biologist counted the number of bacteria cells growing in a culture every hour. The pattern $4, 7, 14, 25, 40, \ldots$ was recorded for the first four hours, with 4 being the initial number present. Show that this sequence contains both a linear and a quadratic element. Find an expression for the number of bacteria after t hours, i.e. find $f(t)$. Using your expression for $f(t)$, and trial and error, find in which hour the colony will have reached 529.

Section 1.8 Solving equations

To solve an equation, we need to find the values of the given variable that satisfy the equation.

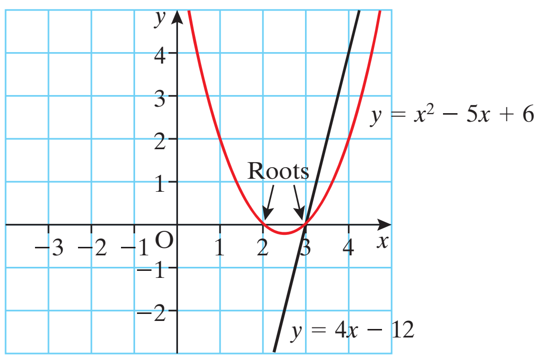
If $4x - 12 = 0$, then $x = 3$ is the only solution of this equation.
If $x^2 - 5x + 6 = 0$, then $x = 2$ and $x = 3$ are both solutions of this equation.
If $y = 4x - 12$, then $(x, y) = (4, 4)$ is one of the many values of (x, y) that satisfy this equation.

Given $f(x) = 4x - 12$,
then $x = 3$ is the value that makes $f(x) = 0$.

Therefore, $(x, y) = (3, 0)$ is the solution of $y = 4x - 12$.

Similarly, $(2, 0)$ and $(3, 0)$ are solutions of $y = x^2 - 5x + 6$.

The values of x that make $y = 0$ are called the **roots** of an equation.



If a graph of the function is plotted, the roots are those points where the graph crosses the x-axis.

Solving linear equations

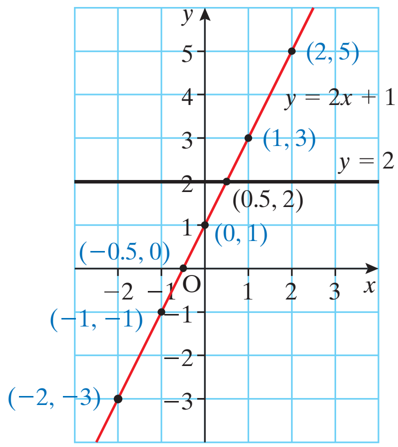
Any equation of the form $y = ax + b$, when plotted, creates a straight line.

Such equations are called **linear equations**.

To solve the equation $2x + 1 = 0$, we need to find the value of x that makes $y = 0$, i.e. where the line crosses the x-axis.

$$2x + 1 = 0$$

$$\Rightarrow \quad x = \frac{-1}{2} = -0.5$$



Similarly, to solve the equation $2x + 1 = 2$, we have to find the value of x that makes $y = 2$, i.e. where the line crosses the line $y = 2$.

$$2x + 1 = 2$$

$$\Rightarrow \quad x = \frac{1}{2} = 0.5$$

In each case, there is only one value of x (one root) produced.

Example 1

Solve the linear equation $\dfrac{2t-3}{5} + \dfrac{1}{20} = \dfrac{t-1}{4}$.

$$\frac{2t-3}{5} + \frac{1}{20} = \frac{t-1}{4}$$

$$\frac{4(2t-3)}{20} + \frac{1}{20} = \frac{5(t-1)}{20} \qquad \text{... finding the lowest common denominator}$$

$$4(2t-3) + 1 = 5(t-1)$$

$$8t - 12 + 1 = 5t - 5 \qquad \text{... expanding}$$

$$3t = 6$$

$$t = 2$$

Exercise 1.8

1. Explain why each of the following is a **non-linear** equation.

(i) $y = 2x^2 + 2x - 1$

(ii) $y = \dfrac{2}{(x-1)} = 2(x-1)^{-1}$

(iii) $y^2 = 3x + 4$

2. Solve each of the following equations.

(i) $5x - 3 = 32$ (ii) $3x + 2 = x + 8$ (iii) $2 - 5x = 8 - 3x$

3. Solve each of these equations.

(i) $2(x - 3) + 5(x - 1) = 3$ (ii) $2(4x - 1) - 3(x - 2) = 14$

(iii) $3(x - 1) - 4(x - 2) = 6(2x + 3)$ (iv) $3(x + 5) + 2(x + 1) - 3x = 22$

4. Solve each of the following equations:

(i) $\frac{2x + 1}{5} = 1$ (ii) $\frac{3x - 1}{4} = 8$ (iii) $\frac{x - 3}{4} = \frac{x - 2}{5}$

5. Find the value of the unknown in each of the following equations:

(i) $\frac{2a}{3} - \frac{a}{4} = \frac{5}{6}$ (ii) $\frac{b + 2}{4} - \frac{b - 3}{3} = \frac{1}{2}$ (iii) $\frac{3c - 1}{6} - \frac{c - 3}{4} = \frac{4}{3}$

6. Find the value of the unknown in each of the following equations:

(i) $\frac{x - 2}{5} + \frac{2x - 3}{10} = \frac{1}{2}$ (ii) $\frac{3y - 12}{5} + 3 = \frac{3(y - 5)}{2}$

(iii) $\frac{3p - 2}{6} - \frac{3p + 1}{4} = \frac{2}{3}$ (iv) $\frac{3r - 2}{5} - \frac{2r - 3}{4} = \frac{1}{2}$

7. Solve each of the following:

(i) $\frac{3}{4}(2x - 1) - \frac{2}{3}(4 - x) = 2$ (ii) $\frac{2}{3}(x - 1) - \frac{1}{5}(x - 3) = x + 1$

Section 1.9 Solving simultaneous linear equations

1. Solving simultaneous linear equations with two variables

The linear equation $y = \frac{2}{3}x - 3$ can be rearranged as follows:

$$y = \frac{2}{3}x - 3$$

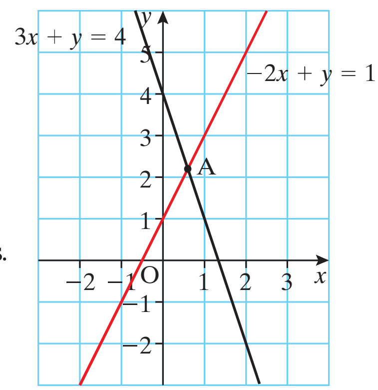
$$3y = 2x - 9$$

$2x - 3y - 9 = 0$ is the equation of the same line, expressed in standard form.

> Standard form of the equation of a line:
>
> $ax + by + c = 0$

This equation has two variables (x, y) for which there are many solutions.
However, if we have two equations in x and y, they are either

(a) parallel, with no point of intersection or
(b) they intersect at a point (x_1, y_1), common to both lines.



In this diagram, the lines $2x - y + 1 = 0$ and $3x + y - 4 = 0$ are plotted.

They have a point of intersection $A(x, y) = \left(\frac{3}{5}, \frac{11}{5}\right)$ which satisfies both equations at the same time, i.e. simultaneously.

We can solve two linear equations (i.e. find their point of intersection) by:

(i) substitution (ii) elimination (iii) graphically (as above).

Example 1

Solve the equations $3x - y = 1$ and $x - 2y = -8$.

Take $x - 2y = -8$ and rearrange the terms, finding ***x* in terms of *y***,

$\rightarrow x = -8 + 2y$

Substitute this expression for x into the second linear equation.

$$\begin{aligned} 3x - y = 1 \text{ becomes } 3(-8 + 2y) - y &= 1 \\ -24 + 6y - y &= 1 \quad \text{... expanding} \\ 5y &= 25 \\ y &= 5 \end{aligned}$$

If $y = 5$, then $x = -8 + 2y$ becomes $x = -8 + 2(5) = 2$

Therefore $(x, y) = (2, 5)$ is the point of intersection.

[Note: Since $3(2) - (5) = 1$ and $(2) - 2(5) = -8$, the point $(2, 5)$ is on both lines.]

Note 1: The technique for solving equations by substitution will be used later to find the point(s) of intersection of a line and a curve.

Note 2: Either variable can be substituted.
(Always choose the variable that is easiest to isolate.)

Example 2

Solve the equations $2x - 5y = 9$ and $3x + 2y = 4$.

$$\begin{aligned} &\text{Let A be } 2x - 5y = 9 \\ &\text{Let B be } 3x + 2y = 4 \\ \therefore \quad &3A: 6x - 15y = 27 \\ \text{and} \quad &2B: \underline{6x + 4y = 8} \\ &-19y = 19 \quad \text{... Subtracting to } \underline{\text{eliminate}}\ x \\ &y = -1 \end{aligned}$$

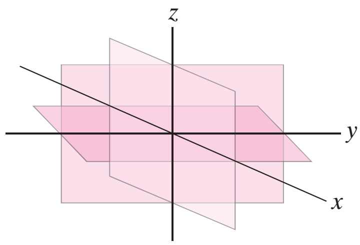
Now substituting $y = -1$ into A, we get
$$\begin{aligned} 2x - 5(-1) &= 9 \\ 2x &= 4 \\ x &= 2 \end{aligned}$$

The point of intersection is $(x, y) = (2, -1)$.

[Note: $2(2) - 5(-1) = 9$ and $3(2) + 2(-1) = 4$ and so this point is on both lines.]

2. Solving simultaneous equations with three variables

$x + y + z = 6$ is an equation with three variables (three dimensions).



To plot this equation we need three axes, x, y and z-axes, at right angles to one another.

When plotted, this equation represents a plane of points.

If three planes are plotted on the same axes, there will be one point of intersection $(x, y, z,)$ provided none of the planes are parallel.

To solve an equation with three unknowns,

(i) reduce the three equations to two by eliminating one of the unknowns
(ii) choose an unknown that is easy to isolate
(iii) eliminate this unknown from all three equations, taking them two at a time
(iv) solve these two equations as before
(v) use your values for these unknowns in one of the original equations to solve for the third unknown
(vi) check your solutions in all equations.

Example 3

Solve the simultaneous equations:

$$\begin{aligned} A:&\quad x + y + z = 6 \\ B:&\quad 2x + y - z = 1 \\ C:&\quad 4x - 3y + 2z = 4 \end{aligned}$$

Eliminating z from all three equations we get,

$$\begin{array}{ll} A: & x + y + z = 6 \\ \underline{B:} & \underline{2x + y - z = 1} \\ D: & 3x + 2y \qquad = 7 \end{array}$$ ... adding A and B

$$\begin{array}{ll} 2B: & 4x + 2y - 2z = 2 \\ \underline{C:} & \underline{4x - 3y + 2z = 4} \\ E: & 8x - y \qquad = 6 \end{array}$$ ... adding 2B and C

$$\begin{array}{ll} D: & 3x + 2y = 7 \\ \underline{2E:} & \underline{16x - 2y = 12} \\ & 19x \qquad = 19 \end{array}$$ ... adding D and 2E

$$x = 1$$

$\therefore$ Using D: $3(1) + 2y = 7$

$$2y = 4$$
$$y = 2$$

Finally, using A: $(1) + (2) + z = 6$

$$z = 3$$

$\therefore$ the point of intersection is $(x, y, z) = (1, 2, 3)$

[**Note:** A: $(1) + (2) + (3) = 6$, and
B: $2(1) + (2) - (3) = 1$, and
C: $4(1) - 3(2) + 2(3) = 4$ and so this point satisfies all three equations]

3. Simultaneous equations in context

Example 4

An opera was attended by 240 people. Two ticket prices, €31 and €16, were available. If the total takings on the night were €5595, find using this data

(i) two linear equations connecting the two types of tickets sold
(ii) the number of €31 tickets sold
(iii) the number of €16 tickets sold.

Let x represent the number of €16 tickets sold and y represent the number of €31 tickets sold.

(i) Since 240 people attended altogether, then A: $x + y = 240$
If the receipts were €5595, then B: $16x + 31y = 5595$

(ii) Solving, we get

$$\begin{aligned} 16\text{A}: 16x + 16y &= 3840 \\ \underline{\text{B}: 16x + 31y} &\underline{= 5595} \\ -15y &= -1755 \quad \text{(subtracting)} \\ y &= 117, \text{ i.e. 117 €31 tickets were sold} \end{aligned}$$

(iii) Using $y = 117$: $x + (117) = 240$
$x = 123$, i.e. 123 €16 tickets were sold.

Example 5

Fifty, twenty and ten cent coins are collected from a coin machine and counted. The total value of the coins is €32. When counting, the cashier noted that twice the number of twenty cent coins, added to the number of ten cent coins, equalled three times the number of fifty cent coins. She then noticed that four times the number of fifty cent coins, added to the number of ten cent coins, equalled six times the number of twenty cent coins.
Find the number of each type of coin in the machine.

Let x = the number of 50 cent coins
Let y = the number of 20 cent coins
Let z = the number of 10 cent coins.

(i) $50x + 20y + 10z = 3200$ … €32 = 3200c
(ii) $2y + z = 3x$
(iii) $4x + z = 6y$

Rearranging the equations into standard form, we get
A: $50x + 20y + 10z = 3200$
B: $3x - 2y - z = 0$
C: $4x - 6y + z = 0$

Adding B and C eliminates z $\quad \therefore$ B + C $\Rightarrow 7x - 8y = 0$.

Adding A and 10B also eliminates z (and y in this case) $\therefore$ A: $50x + 20y + 10z = 3200$

$$\underline{10\text{B}: 30x - 20y - 10z = 0}$$

$$\text{A} + 10\text{B} \Rightarrow 80x = 3200$$

$$\therefore\ x = 40$$

Since $7x - 8y = 0 \Rightarrow 7(40) - 8y = 0$

$\Rightarrow 280 - 8y = 0$

$\Rightarrow y = 35$

Also, $3x - 2y - z = 0 \Rightarrow 3(40) - 2(35) - z = 0 \quad \therefore\ z = 120 - 70 = 50.$

$\therefore\ (x, y, z) = (40, 35, 50).$

There are forty 50c coins, thirty-five 20c coins and fifty 10c coins in the machine.

Exercise 1.9

1. Find the point of intersection of each of the following pairs of lines.

(i) $3x - 2y = 8$
$x + y = 6$

(ii) $3x - y = 1$
$x - 2y = -8$

(iii) $2x - 5y = 1$
$4x - 3y - 9 = 0$

2. Solve each of the following pairs of simultaneous equations.

(i) $4x - 5y = 22$
$7x + 3y - 15 = 0$

(ii) $\frac{x}{2} - \frac{y}{6} = \frac{1}{6}$
$x - 2y = -8$

(iii) $\frac{4x - 2}{5} = \frac{8y}{10}$
$18x - 20y = 4$

3. Solve for x and y given that $\frac{2x - 5}{3} + \frac{y}{5} = 6$ and $\frac{3x}{10} + 2 = \frac{3y - 5}{2}$.

4. Given that $y = 3x - 23$ and $y = \frac{x}{2} + 2$, find the values of y and x.

5. Solve the following equations with three unknowns.

(i) $2x + y + z = 8$
$5x - 3y + 2z = 3$
$7x + y + 3z = 20$

(ii) $2x - y - z = 6$
$3x + 2y + 3z = 3$
$4x + y - 2z = 3$

(iii) $2x + y - z = 9$
$x + 2y + z = 6$
$3x - y + 2z = 17$

6. Find the point of intersection of each of the following sets of planes.

(i) $2a + b + c = 8$
$5a - 3b + 2c = -3$
$7a - 3b + 3c = 1$

(ii) $x + y + 2z = 3$
$4x + 2y + z = 13$
$2x + y - 2z = 9$

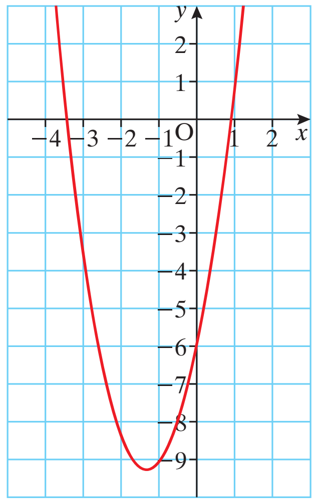
(iii) $x + y + z = 2$
$2x + 3y + z = 7$
$\frac{x}{2} - \frac{y}{6} + \frac{z}{3} = \frac{2}{3}$

7. Find the solution (x, y, z) for

$6x + 4y - 2z - 5 = 3x - 2y + 4z + 10 = 5x - 2y + 6z + 13 = 0.$

8. The curve $f(x) = y = ax^2 + bx + c$ passes through the three points $(1, 2)$, $(2, 4)$ and $(3, 8)$. Use these points to find three equations in a, b and c and hence solve to find $f(x)$.

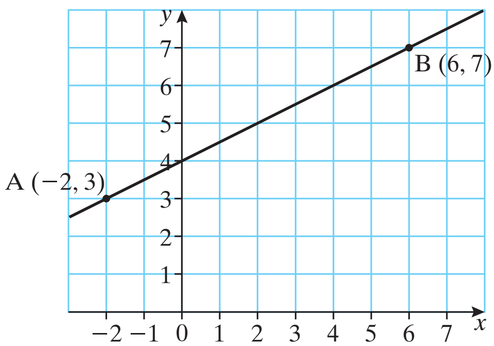
9. A curve of the form $f(x) = y = ax^2 + bx + c$ is drawn as shown.
By using any three points on the curve, form three equations connecting the coefficients a, b and c and hence solve to find $f(x)$.



10. 44 000 people attended a match in Croke Park. The two ticket prices on the day were €30 and €20. The total receipts for the game came to €1.2 million.
How many people paid the higher ticket price?

11. Three years from now, Callum will be twice as old as Lydia was five years ago.
At the moment, half their combined ages is 16. Find their ages.

12. Find the equation of the line AB in the form, $y = ax + b$, by forming two simultaneous equations in a and b using the two given points on the line.
Verify that your line passes through a selected point between A and B.



13. Two forces, N_1 and N_2, act on a hemisphere at rest.
If $\frac{1}{4}N_1 - N_2 = 0$ and $N_1 + \frac{1}{2}N_2 - 99 = 0$,
find the values of N_1 and N_2.

14. If $\dfrac{a}{x-2} + \dfrac{b}{x+2} = \dfrac{4}{(x-2)(x+2)}$ for all values of x,
using algebraic identities, write down two equations in terms of a and b only.
Hence solve for a and b.
Verify your answers using the original equation.

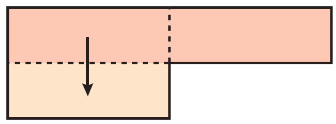
15. If $\dfrac{c}{z-3}+\dfrac{d}{z+2}=\dfrac{4}{(z-3)(z+2)}$ for all values of z, solve for c and d.
Verify your answers by substitution.

16. How many litres of 70% alcohol need to be added to 50 litres of 40% alcohol to make a 50% solution?

17. The sum of two numbers is 26.
If five times the smaller number is taken from four times the larger number, the result is 5.
Let x be the bigger number and y the smaller number.
(i) Write down two equations in x and y.
(ii) Solve these equations for x and y.
(iii) Verify your answers by substitution.

18. A student studied a car rolling down an inclined plane and took measurements of its speed at two different times.
After 7 seconds, it had a speed of 2 m/sec and after 13 seconds, the speed increased to 5 m/sec.
Using the equation $v = u + at$, where v is the speed and t is the time, write down two linear equations in u and a.
Solve these equations to find values for u and a.

19. A farmer builds a long narrow pen for sheep using 60 m of fencing.
If he doubles the width and halves the length, he only needs to use 42 m of fencing.
Find the dimensions of the two pens.



(As can be seen from the diagram, the areas of the two pens remain the same.
Explain why less fencing is needed in the second pen.)

20. The curve $y = ax^2 + bx + c$ contains the points (0, 1), (2, 9) and (4, 41).
(i) Using these points, write three simultaneous equations in a, b and c.
(ii) Hence solve the equations to find the values of a, b and c.

21. Solve the simultaneous equations.

(i) $y - z = 3$
$x - 2y + z = -4$
$x + 2y = 11$

(ii) $\dfrac{x}{3}+\dfrac{y}{2}-z=7$
$\dfrac{x}{4}-\dfrac{3y}{2}+\dfrac{z}{2}=-6$
$\dfrac{x}{6}-\dfrac{y}{4}-\dfrac{z}{3}=1$

22. The circle $x^2 + y^2 + ax + by + c = 0$ passes through the points (1, 0), (1, 2) and (2, 1).
Find the values of a, b, and c.

Revision Exercise 1 (Core)

1. Simplify each of the following algebraic expressions.

(i) $\dfrac{12m^2n^3}{(6m^4n^5)^2}$ (ii) $\dfrac{3+\dfrac{1}{x}}{\dfrac{5}{x}+4}$ (iii) $\dfrac{2+\dfrac{x}{2}}{x^2-16}$

2. Solve for x and y:

(i) $y = x + 4$
$5y + 2x = 6$

(ii) $3x + y = 7$
$x^2 + y^2 = 13$

3. Using long division, find $x^3 - x^2 - 7x + 3 \div x - 3$.

4. Divide $3x^4 - 9x^2 + 27x - 66$ by $x - 2$.

5. Solve the equations.

(i) $x^4 - 9x^2 = 0$ (ii) $(2x - 1)^3(2 - x) = 0$

6. Given that $4x^2 + 20x + k$ is a perfect square, find k.

7. Find the integers a and b such that

(i) $(3 - \sqrt{2})^2 = a - b\sqrt{2}$ (ii) $\left(\dfrac{1-\sqrt{2}}{1+\sqrt{2}}\right) = a\sqrt{2} - b$.

8. Factorise $x^3 - 27$.

9. If $p(x - q)^2 + r = 2x^2 - 12x + 5$ for all values of x, find the values of p, q and r.

10. Solve the simultaneous equations

$$3x + 5y - z = -3$$
$$2x + y - 3z = -9$$
$$x + 3y + 2z = 7.$$

11. Simplify $(b + 1)^3 - (b - 1)^3$.

12. Find the rule (i.e. the nth term) for each of the following quadratic patterns.

(i) 3, 12, 27, 48, 75 …
(ii) 5, 20, 45, 80, 125 …
(iii) 0.5, 2, 4.5, 8, 12.5 …

13. Find the rule for the pattern 6, 12, 20, 30, 42 using first and second differences. Hence find the 100th term of this pattern.

14. Three times the width of a certain rectangle exceeds twice the length by 3 cm. Four times the length is 12 cm more than its perimeter. Find the dimensions of the rectangle.

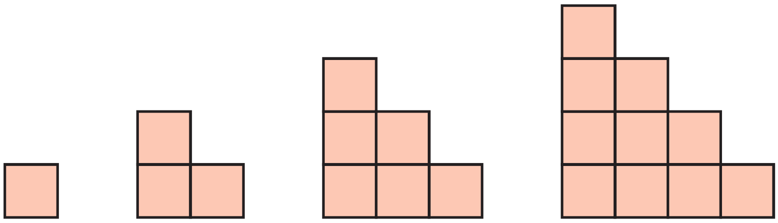
15. The formula for a spherical mirror of radius r cm is given by $\frac{1}{u} + \frac{1}{v} = \frac{2}{r}$, where u cm is the object distance and v cm is the image distance to the mirror.

The magnification in the mirror is given by $m = \frac{v - r}{r - u}$.

(i) Find r in terms of u and v.
(ii) Find m in terms of v and u only.

Revision Exercise 1 (Advanced)

1. By converting the number of squares in the following designs into a number pattern, write down a rule for the pattern. Use the rule to find out how many bricks are needed to build the 49th design.



2. How much soil containing 55% sand needs to be added to 1 m^3 of soil containing 25% sand to make soil containing 35% sand?

Hint: let x m^3 be the amount of soil needed.

3. A metallurgist needs to make 8.4 kg of an alloy containing 50% gold. He is going to melt two metal alloys, one containing 60% gold with a second metal alloy that contains 40% gold.
   (i) Let x kg and y kg be the amounts needed of each metal alloy.
   Write two equations linking the unknowns x and y.
   (ii) Solve the equations to find the amount of each metal needed.

4. If, for all values of x, $(3p - 2t)x + r - 4t^2 = 0$, show that $r = 9p^2$.

5. Simplify the equation $\frac{x + y^2}{x^2} + \frac{x - 1}{x} = -1$ and hence find the ratio of x^2 to y^2.

6. In a chemistry class, a group of students need a 15% acid solution to complete a test.
   The lab only has 10% acid solution and 30% acid solution.
   The students decide to mix the two solutions to get the 15% solution they require.
   If the students need 10 litres of the new solution, find
   (i) the number of litres of the 10% solution they require
   (ii) the number of litres of the 30% solution they require.

7. Brian and Luke race over 50 metres. Brian runs so that it takes him a seconds to run 1 metre. Luke runs so that it takes him b seconds to run 1 metre. Luke wins the race by 1 second. The next day, they race again over 50 metres (and again at the same speeds) but Luke gives Brian a 3-metre start so that Brian only runs 47 metres. Luke wins this race by 0.1seconds. Find
   (i) the values of a and b (ii) Luke's speed.

Revision Exercise 1 (Extended-Response Questions)

1. A football club wanted to organise a family day as a fund-raiser.
They decided to pre-sell tickets at €5.00 for adults and €2.50 for children aged 6 years or younger.
Last year, when they held a similar event 13 000 attended.
Last year they had only one ticket-price.
The organisers wanted to estimate the expected revenue for the day. They decided to use the information obtained from the pre-sold tickets to arrive at this estimate.
However, the members selling the tickets did not record the numbers of adult and children separately. It was known however that 548 tickets in total were sold and that €2460 was collected.

(a) By setting up suitable equations, find
  (i) the number of adult tickets pre-sold
  (ii) the number of children tickets pre-sold
  (iii) the proportion of adult tickets sold.

(b) Based on the same attendance for this year, estimate the revenue expected for the coming fund-raiser.

2. A factory makes two types of sofa. The standard sofa requires 2 hours of work in the manufacturing section and 1 hour in the finishing section.
The deluxe sofa requires 2.5 hours in the manufacturing section and 1.5 hours of finishing work.
Each day, there is a maximum of 48 hours of worker-time available in the manufacturing section and a maximum of 26 hours available in the finishing section.

(i) If x standard sofas and y deluxe sofas are made per day, and the manufacturing section is worked to its capacity, explain why $2x + 2.5y = 48$.
(ii) Find a second equation in x and y if the finishing section is also used to its capacity.
(iii) How many of each sofa can be produced if each section is used to its capacity?

3. A closed rectangular box has a square base of length x cm and height h cm.
The volume of the box is $40\,\text{cm}^3$.

(i) Write an expression for h in terms of x.
(ii) Show that the surface area, $S\,\text{cm}^2$, of this box is given by

$$S = 2x^2 + \frac{160}{x}.$$

(iii) Sketch a graph of S against x, for $0 \leqslant x \leqslant 10$.
(iv) Estimate from the graph the possible values of x and h for which this box has a surface area of $72\,\text{cm}^2$.

4. A game made by a company sells for €11.50. The cost of production consists of an initial cost of €3500 and then €10.50 for each game produced.
Let x be the number of games produced.

(i) If $C(x)$ is the cost of producing x games, find an expression for $C(x)$ in terms of x.
(ii) If $I(x)$ represents the income received for selling x games, find an expression for $I(x)$ in terms of x.

(iii) Plot the graphs of $I(x)$ and $C(x)$ on the same axes. (Scale the x-axis in units of 500 and the y-axis in units of 10,000.)
(iv) How many games need to be sold to recoup the production costs?
(v) Let $P = I - C$. What does P represent?
(vi) How many games need to be sold to make a profit of €2000?

5. Celine has 15 days to complete her quilt for sale at the country market. She can sew blue squares in the quilt at a rate of 4 squares per day and white squares at a rate of 7 squares per day. The quilt is to have a total of 96 squares. The blue fabric costs €0.80 per square and the white fabric costs €1.20 per square.

(a) Find the cost of the quilt.
(b) The 96 squares are used to form a rectangle whose length and width are to be in the ratio $3:2$.
Celine decides to have a rectangle of blue squares in the centre of the quilt surrounded by white squares.
Draw an arrangement of blue and white squares that creates a symmetrical design.

6. A small company manufactures wheelbarrows for the garden-supply market.
The company has overheads of €30 000 per year.
It costs €40 to manufacture each wheelbarrow.
(i) Write a rule which determines the total cost, €C, of manufacturing x wheelbarrows per year.
(ii) If the average production is 6000 wheelbarrows per year, what is the overall cost per wheelbarrow?
(iii) How many wheelbarrows must be made so that the average cost is €46 per wheelbarrow?
(iv) The wheelbarrows are sold to retailers at €80 each. Write a rule which determines the revenue, €R, from the sale of x wheelbarrows to the retailers.
(v) Plot the graphs for C and R on the same axes, with the number of wheelbarrows, x, on the horizontal axis.
(vi) What is the minimum number of wheelbarrows that have to be sold to make a profit each year?
(vii) Write a rule that determines the profit, €P, from the manufacture and sale of x wheelbarrows.

7. Sean joins the end of a long queue to get into a concert. Every time one customer is admitted, Sean skips forward (a) two places or (b) three places.

By completing the following table, form a pattern of numbers for the number of people admitted before Sean, based on the length of the queue.

Using the pattern formed, find the number of people admitted before Sean if there are 70 people in the queue when he joins.

Number of people in the queue before Sean joined	(a) Number of people admitted before Sean. Skipping two	(b) Number of people admitted before Sean. Skipping three
4		
5		
6		
7		
8		
9		
10		
11		
12		
13		
14		
15		
70		