If $z = r(\cos\theta + i\sin\theta)$, then using de Moivre's Theorem:

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta) \text{ for all } n \in Z.$$

Example 1

Find the value of $\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^3$.

$r=1$

$\theta = \frac{\pi}{6}$

$(r\,\text{Cis}\,\theta)^n = r^n\,\text{Cis}\,n\theta$

$$\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^3$$

$$= \cos\frac{\cancel{3}\pi}{\cancel{6}2} + i\sin\frac{\cancel{3}\pi}{\cancel{6}2}$$

$$= \text{Cis}\,\frac{\pi}{2}$$

Exercise 3.9

1. Use de Moivre's theorem to simplify each of the following, expressing your answers in the form $a + bi$:

(i) $\left(\cos\left(\frac{\pi}{8}\right) + i\sin\left(\frac{\pi}{8}\right)\right)^4$ (ii) $\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)^7$

de Moivre

$(r\,\text{Ci}\,S\,\theta)^n = r^n\,\text{Ci}\,S\,n\theta$

(i) $\left(\text{Cis}\,\frac{\pi}{8}\right)^4 = \text{Cis}\,\frac{\cancel{4}\pi}{\cancel{8}2} = \text{Cis}\,\frac{\pi}{2}$

$= 0 + 1i$

use calculator to change to rectangular form

(ii) $\left(\text{Cis}\,\frac{\pi}{6}\right)^7 = \text{Cis}\,\frac{7\pi}{6}$

$= -\frac{\sqrt{3}}{2} - \frac{1}{2}i$

use calculator to change to rectangular form

2. Given $z = \sqrt{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, express z^4 in the form $a + bi$.

de Moivre

$(r\,\mathrm{Cis}\,\theta)^n = r^n\,\mathrm{Cis}\,n\theta$

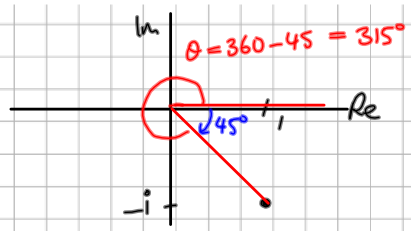
$$z^4 = \left[\sqrt{2}\,\mathrm{Cis}\,\frac{\pi}{3}\right]^4$$
$$= \sqrt{2}^4\,\mathrm{Cis}\,\frac{4\pi}{3}$$
$$= 4\left[\cos\frac{4\pi}{3} + i\,\mathrm{Sin}\,\frac{4\pi}{3}\right]$$
$$= 4\left[-\frac{1}{2} - \frac{\sqrt{3}}{2}\,i\right]$$
$$= -2 - 2\sqrt{3}\,i$$

6. Change each of the following to polar form and then use de Moivre's theorem to express your answers in the form $a + bi$: (i) $(1 - i)^4$

Step 1 : Write in polar form

Polar : $r\,\mathrm{Cis}\,\theta$

$r = \sqrt{a^2 + b^2}$



$\theta = 315°$

$r = \sqrt{1^2 + 1^2}$ $r = \sqrt{2}$

Polar: $1 - i = \sqrt{2}\,\mathrm{Cis}\,315°$

Step 2 : use de Moivre's theorem to expand

$(r\,\mathrm{Cis}\,\theta)^n = r^n\,\mathrm{Cis}\,n\theta$

$$(1-i)^4 = \left(\sqrt{2}\,\mathrm{Cis}\,315°\right)^4 = \sqrt{2}^4\,\mathrm{Cis}\,4(315°)$$
$$= 2\,(\mathrm{Cis}\,1260°)$$
$$= 2\,(-1 + 0i)$$
$$= -2$$

Step 3 : use calculator to convert back to rectangular form.

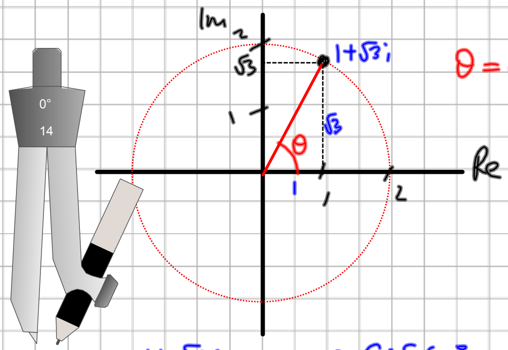
6. Change each of the following to polar form and then use de Moivre's theorem to express your answers in the form $a + bi$: (ii) $(1+\sqrt{3}i)^3$

Step 1 : Write in polar form

Polar : $r\,\text{CiS}\,\theta$

$r = \sqrt{a^2+b^2}$

$r = \sqrt{1^2+\sqrt{3}^2} = \sqrt{4} = 2$



$\theta = 60°$

$1+\sqrt{3}i = 2\,\text{CiS}\,60°$

Step 2 : use deMoivre's theorem to expand

$(r\,\text{CiS}\,\theta)^n = r^n\,\text{CiS}\,n\theta$

$(1+\sqrt{3}i)^3 = (2\,\text{cis}\,60°)^3 = 2^3\,\text{CiS}\,3(60°)$

$= 8\,\text{CiS}\,180°$

$= 8(-1+0i)$

$= -8$

Step 3 : use calculator to convert back to rectangular form.