Solutions to
Test on Differentiation

Questions:
2014 Paper 1 Q. 4 (25 marks)
2012 Paper 1 Q. 6 (25 marks)

Leaving Cert Higher Level Maths
7 October 2014

Question 4 **(25 marks)**

(a) Differentiate the function $2x^2 - 3x - 6$ with respect to x from first principles.

use :

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f(x) = 2x^2 - 3x - 6$$

$$\begin{aligned} f(x+h) &= 2(x+h)^2 - 3(x+h) - 6 \\ &= 2[x^2 + 2xh + h^2] - 3x - 3h - 6 \\ &= 2x^2 + 4xh + 2h^2 - 3x - 3h - 6 \end{aligned}$$

$$\begin{aligned} \frac{f(x+h)-f(x)}{h} &= \frac{4x\cancel{h} + 2h^{\cancel{2}} - 3\cancel{h}}{\cancel{h}} \\ &= 4x + 2h - 3 \end{aligned}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = 4x + 0 - 3$$

$$f'(x) = 4x - 3$$

(b) Let $f(x) = \frac{2x}{x+2}$, $x \neq -2$, $x \in \mathbb{R}$. Find the co-ordinates of the points at which the slope of the tangent to the curve $y = f(x)$ is $\frac{1}{4}$.

Quotient Rule

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$u = 2x$, $\frac{du}{dx} = 2$

$v = x+2$, $\frac{dv}{dx} = 1$

Points when $f'(x) = \frac{1}{4}$?

solve

$x = -6$, $y = ?$

$x = 2$, $y = ?$

$$\begin{aligned} f'(x) &= \frac{(x+2)(2) - (2x)(1)}{(x+2)^2} \\ &= \frac{2x + 4 - 2x}{(x+2)^2} \end{aligned}$$

$$\boxed{f'(x) = \frac{4}{(x+2)^2}}$$

$$\Rightarrow \frac{4}{(x+2)^2} = \frac{1}{4}$$

$$\begin{aligned} 16 &= (x+2)^2 \\ 16 &= x^2 + 4x + 4 \\ x^2 + 4x - 12 &= 0 \\ (x+6)(x-2) &= 0 \\ x = -6, &\quad x = 2 \end{aligned}$$

$y = \frac{2(-6)}{-6+2} = \frac{-12}{-4} = 3$ $\boxed{\text{pt. } (-6, 3)}$

$y = \frac{2(2)}{2+2} = \frac{4}{4} = 1$ $\boxed{\text{pt. } (2, 1)}$

Question 6 **(25 marks)**

(a) Let $f(x) = e^{-\frac{1}{2}x^2}$.

Show that the second derivative of $f(x)$ with respect to x is $f''(x) = (x^2 - 1)e^{-\frac{1}{2}x^2}$.

$f(x) \to f'(x)$

$e^x \to e^x$

CHAIN RULE

$f'(x) = \left(e^{-\frac{1}{2}x^2}\right).\left(-\frac{\cancel{2}x}{\cancel{2}}\right)$

$f'(x) = -xe^{-\frac{1}{2}x^2}$

PRODUCT RULE

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$u = -x \Rightarrow \frac{du}{dx} = -1$

$v = e^{-\frac{1}{2}x^2} \Rightarrow \frac{dv}{dx} = -xe^{-\frac{1}{2}x^2}$

as in part i

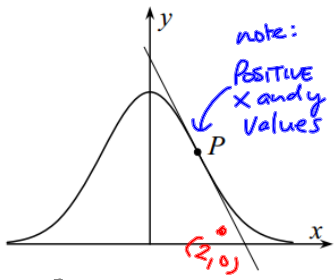
$f''(x) = (-x)(-xe^{-\frac{1}{2}x^2}) + (e^{-\frac{1}{2}x^2})(-1)$

$= x^2e^{-\frac{1}{2}x^2} - e^{-\frac{1}{2}x^2}$

$f''(x) = (x^2 - 1)e^{-\frac{1}{2}x^2}$

QED

(b) The point P in the first quadrant is a point of inflection of the curve $y = e^{-\frac{1}{2}x^2}$.
Show that the tangent at P crosses the x-axis at $(2, 0)$.



FROM PART a

at inflection $f''(x) = 0$

$f''(x) = (x^2 - 1)e^{-\frac{1}{2}x^2}$

$\Rightarrow (x^2 - 1)e^{-\frac{1}{2}x^2} = 0$

$\Rightarrow x^2 - 1 = 0$ or $e^{-\frac{1}{2}x^2} = 0$ not possible!

$x^2 = 1$

$\boxed{x = \pm 1}$

P is in first quadrant $\Rightarrow$ x is positive

$x = 1$, $y = ?$

$\Rightarrow x = 1$

$y = e^{-\frac{1}{2}(1)^2} = e^{-\frac{1}{2}} \Rightarrow \boxed{P(1, e^{-\frac{1}{2}})}$

Slope at P ?

$f'(x) = -xe^{-\frac{1}{2}x^2}$

$f'(1) = -1e^{-\frac{1}{2}(1)^2} = -e^{-\frac{1}{2}}$

Show slope between $P(1, e^{-\frac{1}{2}})$ and $(2, 0)$ is $-e^{-\frac{1}{2}}$

$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{e^{-\frac{1}{2}} - 0}{1 - 2} = -e^{-\frac{1}{2}} \Rightarrow$ Colinear