chapter 3

Probability 2

Section 3.3 Binomial distribution – Bernoulli trials

PROJECT MATHS
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A coin is biased in such a way that the probability of a head is always $\frac{2}{5}$.

Robbie tosses the coin four times. He wants to know the probability that there will be three heads and one tail.

The 3 heads and 1 tail can be arranged in **four** different ways.

$$P(\text{H}, \text{H}, \text{H}, \text{T}) = \tfrac{2}{5} \times \tfrac{2}{5} \times \tfrac{2}{5} \times \tfrac{3}{5} = \left(\tfrac{2}{5}\right)^3 \times \tfrac{3}{5}$$

$$P(\text{H}, \text{H}, \text{T}, \text{H}) = \tfrac{2}{5} \times \tfrac{2}{5} \times \tfrac{3}{5} \times \tfrac{2}{5} = \left(\tfrac{2}{5}\right)^3 \times \tfrac{3}{5}$$

$$P(\text{H}, \text{T}, \text{H}, \text{H}) = \tfrac{2}{5} \times \tfrac{3}{5} \times \tfrac{2}{5} \times \tfrac{2}{5} = \left(\tfrac{2}{5}\right)^3 \times \tfrac{3}{5}$$

$$P(\text{T}, \text{H}, \text{H}, \text{H}) = \tfrac{3}{5} \times \tfrac{2}{5} \times \tfrac{2}{5} \times \tfrac{2}{5} = \left(\tfrac{2}{5}\right)^3 \times \tfrac{3}{5}$$

The total probability for 3 heads and 1 tail $= 4 \times \left(\frac{2}{5}\right)^3 \times \frac{3}{5} = \frac{96}{625}$

Notice that the 4 in the answer is the value of $\binom{4}{3}$ and is the number of selections of 3 heads from 4 coins.

Thus the probability of 3 heads and 1 tail $= \binom{4}{3}\left(\frac{2}{5}\right)^3 \times \frac{3}{5}$

The example above is a special type of probability model called the **binomial distribution**.

A binomial distribution can be used in any experiment that has these 4 characteristics:

- A fixed number, n, of trials are carried out
- Each trial has two possible outcomes: success or failure
- The trials are independent
- The probability of success in each trial is constant.
  The probability of a success is generally called p.
  The probability of a failure is q, where $p + q = 1$.

In general, the probability of r successes in n trials is given by the formula on the right, where p is the probability of success and q is the probability of failure.

$$P(r \text{ successes}) = \binom{n}{r} p^r q^{n-r}$$

Experiments which satisfy the four conditions listed above are also called **Bernoulli Trials** after the Swiss mathematician James Bernoulli (1654–1705).

Consider the event of obtaining a 6 from a single throw of an unbiased die.

$$P(\text{success}) = \tfrac{1}{6} \quad \text{and} \quad P(\text{failure}) = \tfrac{5}{6}$$

If there are 8 such trials, then the probability of 0, 1, 2, 3, … successes from 8 attempts is given by the terms of the expansion of

$$\left(\tfrac{5}{6} + \tfrac{1}{6}\right)^8$$

Since the probability of r successes is given by $\binom{n}{r} p^r q^{n-r}$

(i) $P(\text{no six}) = \binom{8}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^8 = \binom{8}{0}\left(\frac{5}{6}\right)^8$

(ii) $P(1 \text{ six}) = \binom{8}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^7$

(iii) $P(2 \text{ sixes}) = \binom{8}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^6$

....

$P(8 \text{ sixes}) = \binom{8}{8}\left(\frac{1}{6}\right)^8\left(\frac{5}{6}\right)^0 = \binom{8}{8}\left(\frac{1}{6}\right)^8$

$P(r \text{ successes}) = \binom{n}{r} p^r q^{n-r}$

Example 1

An unbiased die is thrown 5 times. Find the probability of obtaining

(i) 1 six (ii) 3 sixes (iii) at least 1 six.

$n = 5$, $r = 1$, $p = 1/6$, $q = 5/6$

(i) $P(1 \text{ six in } 5 \text{ trials}) = \binom{5}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^4 = \frac{3125}{7776}$

$n = 5$, $r = 3$, $p = 1/6$, $q = 5/6$

(ii) $P(3 \text{ sixes in } 5 \text{ trials}) = \binom{5}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 = \frac{125}{3888}$

$n = 5$, $r = 0$, $p = 1/6$, $q = 5/6$

(iii) $P(\text{at least 1 six}) = 1 - P(\text{no six in 5 trials})$

$= 1 - \binom{5}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^5 = \frac{4651}{7776}$

$P(r \text{ successes}) = \binom{n}{r} p^r q^{n-r}$

Example 2

Given that 10% of apples are bad, find the probability that in a box containing 6 apples, there is

(i) no bad apple

(ii) just one bad apple

(iii) at least one bad apple.

$n = 6$, $r = 0$, $p = 1/10$, $q = 9/10$

(i) $P(\text{no bad apple}) = \binom{6}{0}\left(\frac{1}{10}\right)^0\left(\frac{9}{10}\right)^6 = \left(\frac{9}{10}\right)^6$

$n = 6$, $r = 1$, $p = 10\%$, $q = 90\%$

(ii) $P(1 \text{ bad apple}) = \binom{6}{1}(10\%)^1(90\%)^5 = \frac{6 \cdot 9^5}{10^6}$

(iii) $P(\text{at least } 1) = 1 - P(\text{no bad})$

$= 1 - \left(\frac{9}{10}\right)^6$

Probability of k^{th} success on n^{th} Bernoulli trial

In example 1 on the previous page, we worked out the probability of getting 3 sixes when a dice is thrown 5 times. If the same dice is thrown continuously until a six appears for the fourth time, how do we find the probability that the 4th six appears on the tenth throw?

For a 4th six to appear on the 10th throw,

(i) we need to get 3 sixes on the first nine throws, and then

(ii) get a six on the 10th throw.

Three sixes on the first nine throws is given by

$$\binom{9}{3} p^3 q^{9-3} \qquad \text{i.e. } \binom{9}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^6$$

$$84 \times \frac{5^6}{6^9} = 0.13$$

$P(\text{six on the 10th throw}) = \frac{1}{6}$

Thus $P(\text{4th six on the 10th throw}) = 0.13 \times \frac{1}{6} = 0.0217$