Example 4

a How many four-digit numbers can be formed using the digits 0, 2, 5, 7, 8 if a digit cannot be used more than once in any number? 5 digits

b (i) How many of these numbers are greater than 5000?
(ii) How many of these numbers are odd?

a) If we can't start with 0.

$\boxed{4} \times \boxed{4} \times \boxed{3} \times \boxed{2} = 96$
not 0

b) (i) If > 5000 starts with 5, 7 or 8

$\boxed{3} \times \boxed{4} \times \boxed{3} \times \boxed{2} = 72$
5,7,8

(ii) If odd it ends with 5 or 7

$\boxed{3} \times \boxed{3} \times \boxed{2} \times \boxed{2} = 36$
not 0 ... 5,7

4. Permutations of n different objects taking r of them at a time

To find the number of ways the five letters A, B, C, D, E can be arranged in a line when taking 3 at a time, we could use boxes as follows:

$$\boxed{5}\boxed{4}\boxed{3} = 5 \times 4 \times 3 = 60 \text{ ways.}$$

The first box can be filled in 5 ways, the second in 4 ways and the third in 3 ways.

Notice that $5 \times 4 \times 3 = \dfrac{5 \times 4 \times 3 \times 2 \times 1}{2 \times 1} = \dfrac{5!}{2!} = \dfrac{5!}{(5-3)!}$

We use the notation ${}^5P_3$ to denote the number of permutations of 5 objects, taking them 3 at a time.
${}^5P_3 = 5 \times 4 \times 3$ ... starting at 5 and going down 3 numbers

Similarly, ${}^8P_4 = 8 \times 7 \times 6 \times 5 \left(\text{or } \dfrac{8!}{(8-4)!}\right)$

In general, the number of arrangements of n objects, taking r at a time, is given on the right.

$${}^nP_r = \frac{n!}{(n-r)!}$$

Example 5

(i) Evaluate $^{10}P_3$ (ii) Find n if $7[^nP_3] = 6[^{n+1}P_3]$

(i) $10P3 = 720$

$\boxed{10} \times \boxed{9} \times \boxed{8} = 720$

(ii) $7(^nP_3) = 6(^{n+1}P_3)$

$7(\boxed{\cancel{n}} \times \boxed{\cancel{n-1}} \times \boxed{n-2}) = 6(\boxed{n+1} \times \boxed{\cancel{n}} \times \boxed{\cancel{n-1}})$

$7(n-2) = 6(n+1)$

$7n-14 = 6n+6$

$n = 20$

check $7(^{20}P_3) = 47880$

$6(^{21}P_3) = 47880$ ✓

Example 6

a How many different four-letter arrangements can be made from the letters of the word THURSDAY if a letter cannot be repeated in an arrangement?

b How many of the arrangements begin with the letter D and end with a vowel?

a 8 letter

no repeat $= 8! = 40320$

b

$\underset{\text{begin with D}}{\boxed{1}} \times \boxed{6} \times \boxed{5} \times \boxed{4} \times \boxed{3} \times \boxed{2} \times \boxed{1} \times \underset{\text{end vowel}}{\boxed{2}}$

$= 6!(2) = 1440$

6.a In how many ways can seven children sit on a bench?
b In how many of these arrangements are the two oldest always together?

a) $7! = 5040$

b) Treat as if 6 children × 2

$$\underbrace{\boxed{6}\times\boxed{5}\times\boxed{4}\times\boxed{3}\times\boxed{2}\times\boxed{1}}_{\text{2 oldest treated as 1 person}} \times 2$$

2 ways oldest can be arranged

$6! \times 2 = 1440$

12. Three girls and four boys are to sit in a row of seven chairs.
How many different arrangements are possible
(i) if the girls sit beside one another
(ii) if no two boys may sit beside each other?

(i) Treat as 5 chairs × ways of arranging the girls

$\boxed{\square\ \square\ \square} \times \square\ \square\ \square\ \square$

Girls Boys

$5! \times 3! = 720$

(ii) Girls sit between boys

$\boxed{4}\times\boxed{3}\times\boxed{3}\times\boxed{2}\times\boxed{2}\times\boxed{1}\times\boxed{1}$

$4! \times 3! = 144$

17. How many three-digit numbers can be formed using the digits
(i) 1 to 9 (ii) 0 to 9
if a digit cannot be repeated in the same number?

(i) $\boxed{9} \times \boxed{8} \times \boxed{7} = 504$

or $^9P_3 = 504$

(ii) $\boxed{9} \times \boxed{9} \times \boxed{8} = 648$

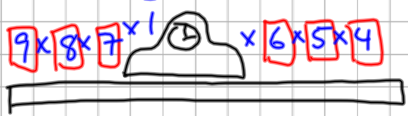
not 0

If 012 is not a 3 digit no.

24. A woman has 10 ornaments, including a clock, of which only 7 will fit on the mantelpiece. If the clock must go in the centre, how many different arrangements can be made with the ornaments?

Clock

$\boxed{9} \times \boxed{8} \times \boxed{7} \times 1 \times \boxed{6} \times \boxed{5} \times \boxed{4}$



$\square\ \square\ \square$ unused

Permutation
How ways of arranging 6 out of 9 choices
9 P C

$^9P_6 = 60480$

25. Seven children, including one set of twins, are arranged in a line.

a How many different arrangements can be made?

b In how many of these arrangements are the twins

(i) always together (ii) always apart?

a Arrange 7 children with no restrictions $= 7! = 5040$

b i Twins always together $= 6! \times 2! = 1440$

Treat as 6 children × ways of arranging the twins

ii Always apart ⇒ twins are not together

Total arrangements: 5040
− Twins together: − 1440
Twins apart 3600