1. Formula for the sum of a series

e.g. prove by induction that

$$\frac{1}{3}+\frac{2}{9}+\frac{4}{27}+\ldots+\frac{2^{n-1}}{3^n}=1-\left(\frac{2}{3}\right)^n.$$

① Show true for $n=1$ & $n=2$

$n=1$: Is $\frac{1}{3} = 1-\left(\frac{2}{3}\right)^1$? yes

$n=2$ Is $\frac{1}{3}+\frac{2}{9} = 1-\left(\frac{2}{3}\right)^2$?

ie. is $\frac{5}{9} = 1-\frac{4}{9}$? yes

② Assume true for $n=k$

Assume: $\left(\frac{1}{3}+\frac{2}{3}+\frac{4}{27}+\ldots+\frac{2^{k-1}}{3^k}\right) = 1-\left(\frac{2}{3}\right)^k$

③ Prove for $n=k+1$

ie. show LHS $= 1-\left(\frac{2}{3}\right)^{k+1}$

If $n=k+1$; LHS $=\left(\frac{1}{3}+\frac{2}{3}+\ldots+\frac{2^{k-1}}{3^k}\right)+\frac{2^{k\cancel{+1-1}}}{3^{k+1}}$

$= \left(1-\left(\frac{2}{3}\right)^k\right)+\frac{2^k}{3^{k+1}} = 1-\left(\frac{2}{3}\right)^k+\frac{1}{3}\left(\frac{2}{3}\right)^k$

$= 1-\frac{2}{3}\left(\frac{2}{3}\right)^k = 1-\left(\frac{2}{3}\right)^{k+1}$

QED

④ Conclude

It is true for $n=1$, $n=2$, $n=k$, $n=k+1$

$\Rightarrow$ it is true for all $n \in \mathbb{N}$.

2. Divisibility proofs

e.g. prove by induction that

$$8^n - 7n + 6$$

is divisible by 7, for all $n \in \mathbb{N}$

① Show true for $n=1$ & $n=2$

$n=1$; $8^1 - 7(1) + 6 = 7 = 7(1)$

$n=2$ $8^2 - 7(2) + 6 = 56 = 7(8)$

② Assume true for $n=k$

Assume $\underline{8^k - 7k + 6} = 7a$, $a \in \mathbb{N}$

③ Prove for $n=k+1$

ie.. show $8^{k+1} - 7(k+1) + 6$ is divisible by 7

$n=k+1$: $8^{k+1} - 7(k+1) + 6$

$= 8.8^k - 7k - 7 + 6$

$= 8.8^k - 7 \quad - 7k + 6$

$= 7.8^k + 8^k - 7 \quad - 7k + 6$

$= 7(8^k - 1) + \underline{8^k - 7k + 6}$

$= 7(8^k - 1) + 7a$

$= 7(8^k - 1 + a)$

QED

④ Conclude

It is true for $n=1$, $n=2$, $n=k$, $n=k+1$

$\Rightarrow$ it is true for all $n \in \mathbb{N}$.

3. Inequality proofs

e.g. prove by induction that

$$2^n > 3n,$$

for all $n \in \mathbb{N}, n \geq 4$.

① Show true for $n=4$ & $n=5$	$n=4$; Is $2^4 > 3(4)$? $16 > 12$ true
② Assume true for $n=k$	$n=5$; Is $2^5 \overset{?}{>} 3(5)$? $32 > 15$ true
③ Prove for $n=k+1$	Assume $2^k > 3k$
ie. Show $2^{k+1} > 3(k+1)$	If $n=k+1$, then is $2^{k+1} > 2k$?
	$LHS = 2^{k+1} = 2.2^k = 2^k + 2^k$ $RHS = 3(k+1) = 3k+3$
	Is $2^k + 2^k > 3k+3$? yes because $2^k > 3k$ and $2^k > 3$ QED
④ Conclude	It is true for $n=1, n=2, n=k, n=k+1$ $\Rightarrow$ it is true for all $n \in \mathbb{N}$.

4. **Prove de Moivre's Theorem** using induction

De Moivre's Theorem states that:	$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
① Show true for $n=1$	$n=1$, $LHS = (\cos\theta + i\sin\theta)^1 = \cos\theta + i\sin\theta$ $RHS = \cos(1)\theta + i\sin(1)\theta = \cos\theta + i\sin\theta$ true
$\cos^2\theta - \sin^2\theta = \cos 2\theta$ $2\sin\theta\cos\theta = \sin 2\theta$	$n=2$, $LHS = (\cos\theta + i\sin\theta)^2$ $= \cos^2\theta + 2i\sin\theta\cos\theta \mp \cancel{i^2}\sin^2\theta$ $= \cos 2\theta + i\sin 2\theta$ $RHS = \cos 2\theta + i\sin 2\theta$ true
② Assume true for $n=k$	Assume: $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$
③ Prove true for $n=k+1$	If $n=k+1$ then $LHS = (\cos\theta + i\sin\theta)^{k+1}$ $= (\cos\theta + i\sin\theta)(\cos\theta + i\sin\theta)^k$ $= (\cos\theta + i\sin\theta)(\cos k\theta + i\sin k\theta)$ $= \cos\theta\cos k\theta + i\cos\theta\sin k\theta + i\sin\theta\cos k\theta \mp \cancel{i^2}\sin\theta\sin k\theta$
$\sin(A+B) = \sin A\cos B + \sin B\cos A$ $\cos(A+B) = \cos A\cos B - \sin A\sin B$	$= \cos(\theta + k\theta) + i\sin(\theta + k\theta)$ $= \cos((k+1)\theta) + i\sin((k+1)\theta) = RHS$ QED
④ Conclude	It is true for $n=1, n=2, n=k, n=k+1$ $\Rightarrow$ it is true for all $n \in \mathbb{N}$.