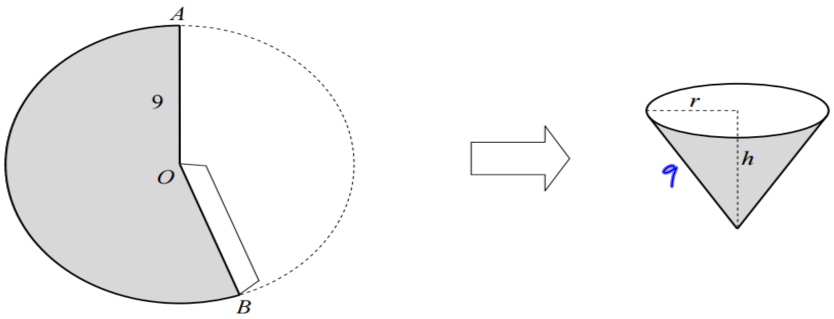
Question 8 **(50 marks)**

A company uses waterproof paper to make disposable conical drinking cups. To make each cup, a sector AOB is cut from a circular piece of paper of radius 9 cm. The edges AO and OB are then joined to form the cup, as shown.

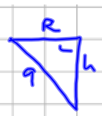
The radius of the rim of the cup is r, and the height of the cup is h.



(a) By expressing r^2 in terms of h, show that the capacity of the cup, in cm^3, is given by the formula

$$V = \frac{\pi}{3}h\left(81-h^2\right).$$

$$\boxed{V = \frac{\pi R^2 h}{3}}$$



$9^2 = R^2 + h^2$
$R^2 = 81 - h^2$

$\Rightarrow \quad V = \frac{\pi\,(81-h^2)h}{3}$ QED

(b) There are two positive values of h for which the capacity of the cup is $\dfrac{154\pi}{3}$.

One of these values is an integer.
Find the two values.
Give the non-integer value correct to two decimal places.

$$V = \frac{\pi}{3}h\left(81-h^2\right).$$

$\Rightarrow \quad \frac{\cancel{\pi}}{\cancel{3}} h\,(81-h^2) = \frac{154\,\cancel{\pi}}{\cancel{3}}$ SOLVE

$81h - h^3 - 154 = 0$

$h^3 - 81h + 154 = 0$

$h = 2 \quad \Rightarrow \quad (2)^3 - 81(2) + 154 = 0 \quad \Rightarrow h = 2$ is a sol^n

DIVIDE By related factor $(h-2)$

$$
\begin{array}{r|l}
 & h^2 + 2h - 77 \\
\hline
h-2 & h^3 \pm 0h^2 - 81h + 154 \\
 & \mp h^3 \pm 2h^2 \\
\hline
 & 2h^2 - 81h \\
 & \mp 2h^2 \pm 4h \\
\hline
 & -77h + 154 \\
 & \pm 77h \mp 154 \\
\hline
 & 0
\end{array}
$$

Solve $h^2 + 2h - 77 = 0$

$\boxed{\begin{array}{l} a = 1 \\ b = 2 \\ c = -77 \end{array}}$

$x_{\text{POSITIVE}} = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \approx 7.83 \quad (2\ \text{dp})$

(c) Find the maximum possible volume of the cup, correct to the nearest cm^3.

at max $\frac{dV}{dh} = 0$

$V = \frac{\pi}{3}h(81 - h^2)$. $= \frac{\pi}{3}[81h - h^3] = 27\pi h - \frac{\pi}{3}h^3$

$\frac{dV}{dh} = 27\pi - \frac{\pi}{\cancel{3}}(\cancel{3}h^2) = 27\pi - h^2\pi$

at max $\Rightarrow$ $27\cancel{\pi} - h^2\cancel{\pi} = 0$

$h^2 = 27$ (h is positive)

$h = \sqrt{27}$

$V_{max} = \frac{\pi}{3}(\sqrt{27})(81 - (\sqrt{27})^2) \approx 294 \text{ cm}^3$