Section B Contexts and Applications 150 marks

Answer **all three** questions from this section.

Question 7 **(50 marks)**

A quantity is subject to exponential growth or decay if it increases or decreases at a rate proportional to its current value. Many physical phenomena such as the growth of human population, the spread of viruses, the rates of radioactive decay, etc. are modelled on exponential functions.

(a) The equation that describes exponential growth is $\frac{dN}{dt} = kN$, which states that the growth rate of the quantity N at time t is proportional to the value $N(t)$.

(i) Show that $N(t) = N_0 e^{kt}$ satisfies the equation $\frac{dN}{dt} = kN$, where N_0 is the quantity at $t = 0$.

$N = N_0 e^{kt} \Rightarrow \frac{dN}{dt} = k[N_0 e^{kt}] = kN$

(ii) Deduce that $\ln N = kt + c$, where c is a constant.

$N = N_0 e^{kt}$

$\ln N = \ln(N_0)(e^{kt}) = \ln N_0 + \ln e^{kt}$
$= \ln N_0 + kt \ln e$
$= \ln N_0 + kt$
$=$ constant $+ kt$

AED

(iii) Suggest one limitation of using exponential functions to model physical phenomenon and give a possible reason for your answer.

Limitation: nature is not following a perfect maths rule.

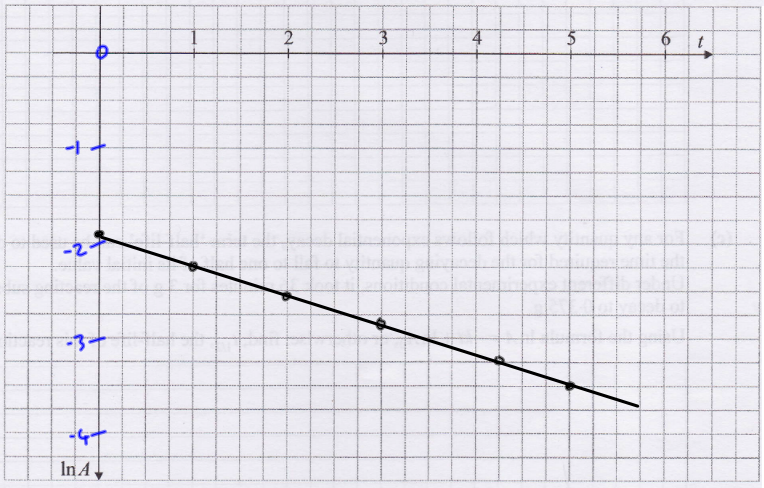
Reason: eg. population growth is complex affected by wealth, education, disease, war...

(b) Exponential decay therefore occurs in the same way when the growth rate is negative. The rates of certain types of chemical reactions, whose rates depend only on the concentration of one or another reacting substance, consequently follow exponential decay. The concentration of the reacting substance is represented by the equation $\ln A = -kt + \ln A_0$, where A_0 is the initial concentration and A is the concentration after time t.

In an experiment to observe the rate of reaction in a chemical process, the concentration of the reacting substance was measured at various times t from when the experiment began. The data is shown in the table below.

t	Time (hours)	0	1	2	3	4	5
A	Concentration (moles litre^{-1})	0·140	0·104	0·075	0·053	0·041	0·030
	ln A	−1·97	−2·26	−2·59	−2·94	−3·19	−3·51

(i) Find, correct to two decimal places, the values of $\ln A$ for each of the concentration measurements in the data set shown and plot $\ln A$ against t on the axes below.



(ii) Using your graph, or otherwise, find the value of k, correct to one decimal place.

use input, output
$t = 1$
$A = 0.104$

$$\ln A = -kt + \ln A_0$$
$$\ln(0.104) = -k(1) + \ln(0.140)$$
$$-2.26 = -k - 1.97$$
$$k \approx 3 \quad (1\text{dp})$$

note: could use slope of graph to find k

(iii) The concentration of the reacting substance when the chemical process is complete was determined to be 0·015 moles litre^{-1}. A
Using your value for k, find the time it took for the experiment to be fully completed, correct to the nearest minute.

$$\ln A = -kt + \ln A_0$$
$$\ln(0.015) = -0.308t - 1.97$$
$$t = \frac{\ln(0.015) + 1.97}{-0.308} = 7.44$$
$$= 7{:}26 \quad \text{h : m}$$

(iv) Show that, according to the exponential model used, the rate of change of $A(t)$ is always increasing over time.

ie Show $\frac{d^2A}{dt^2} > 0$

Use: $k = 0.3$

$\ln A = -kt + \ln A_0$

$\ln A = -0.3t - 1.97$

$A = ?$

$\log_e A = -0.3t - 1.97$ ← power

↑ BASE

If $x^n = y$ then $n = \log_x y$

$A = e^{-0.3t - 1.97}$

Differentiate:

$\frac{dA}{dt} = -0.3\, e^{-0.3t-1.97}$

$\frac{d^2A}{dt^2} = (-0.3)(-0.3)\, e^{-0.3t-1.97}$

$= 0.09\, e^{-0.3t-1.97} > 0$ QED

(c) For any quantity which follows exponential decay, the term 'half-life' can be used to describe the time required for the decaying quantity to fall to one half of its initial value.
Under different experimental conditions, it took 36 minutes for 3 g of the reacting substance to decay to 0·375 g.

Using the formula $\ln A = -kt + \ln A_0$, or otherwise, find, $t_{1/2}$, the half-life of this reaction.

$\ln A = -kt + \ln A_0$

$t = 0$ $A = 3$ g

$t = 36$ $A = 0.375$ g

$\ln 3 = -k(0) + \ln A_0 \Rightarrow A_0 = 3$

$\ln A = -kt + \ln 3$

$\ln(0.375) = -k(36) + \ln 3$

$\ln(0.375) - \ln 3 = -36k$

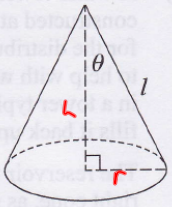
$k = \frac{\ln 3 - \ln(0.375)}{36} = 0.0577$

$t_{\frac{1}{2}}$ = time to reduce to 1.5 g $\ln 1.5 = -0.0577t + \ln 3$

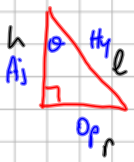
$+t = \frac{\ln 1.5 - \ln 3}{-0.0577} \approx 12$ mins

Question 8 **(50 marks)**

(a) The diagram shows a right circular cone with a fixed slant height of l cm. The slant height makes an angle θ with the vertical, where $0 < \theta < \frac{\pi}{2}$.



(i) Express the radius and height of the cone in terms of l and θ and, hence, show that the volume of the cone is $\frac{\pi l^3}{6}(\sin\theta \sin 2\theta)$.



SOH
CAH
TOA

$\sin\theta = \frac{r}{l}$

$r = l\sin\theta$

$\cos\theta = h/l \Rightarrow h = l\cos\theta$

$V = \frac{1}{3}\pi r^2 h \Rightarrow V = \frac{1}{3}\pi (l\sin\theta)^2 (l\cos\theta)$

$= \frac{\pi}{3} l^3 \sin^2\theta\cos\theta$

using trigonometry $= \frac{\pi l^3}{6}\sin\theta\cos 2\theta$