Number: Logs, Indices, Induction

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Logarithmic Functions

Logarithms were introduced by John Napier in the early 17th century as a means to simplify calculations.

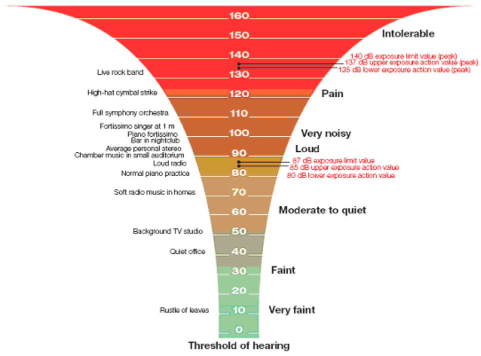
They were rapidly adopted by navigators, scientists, engineers, and others to perform computations more easily, using slide rules and logarithm tables.

Where are Logs used today?

Sound: the decibel is a logarithmic unit quantifying sound pressure and voltage ratios.

Chemistry: pH and pOH are logarithmic measures for the acidity of an aqueous solution. Logarithms are commonplace in scientific formulae.

Where are Logs used today?



Where are Logs used today?

Music: Logs describe musical intervals.
(see: http://www.notreble.com/buzz/2010/02/04/math-and-music-intervals/)

Also, they appear in formulae counting prime numbers, inform some models in psychophysics, and can aid in forensic accounting.

How Do Logs Work?

Logs are directly related to indices

Example:

$$2^3 = 8 \qquad\qquad a^b = c$$

This index form can be re-written in log form:

$$\log_2 8 = 3 \qquad\qquad \log_a c = b$$

How Do Logs Work?

Example:

$$10^2 = 100 \qquad a^b = c$$

This index form can be re-written in log form:

$$\log_{10} 100 = 2 \qquad \log_a c = b$$

How Do Logs Work?

Example:

$$3^5 = 243 \qquad a^b = c$$

This index form can be re-written in log form:

$$\log_3 243 = 5 \qquad \log_a c = b$$

When Might You Use Logs?

Scenario: During your calculations for some Mathematics problem, you end up with the following equation:

$$10^x = 550 \qquad a^b = c$$

This can be solved by converting it into log form:

$$\log_{10} 550 = x \qquad \log_a c = b$$

When Might You Use Logs?

$$\log_{10} 550 = x$$

Use your calculator to solve this log.

$$\log_{10} 550 = 2.74$$

Thus $x = 2.74$

Logs on your Calculator

"log" button refers to $\log_{10}$

"ln" button refers to $\log_e$

What if the base of the log is not 10 or e?

Some calculators can only solve logs that have a base of 10 or e.

If you encounter a log which has a base which is not 10 or e, then you may need to apply the change of base formula:

$$\log_a x = \frac{\log_c x}{\log_c a}$$

What if the base of the log is not 10 or e?

$$\log_a x = \frac{\log_c x}{\log_c a}$$

"c" is the new base you choose (either 10 or e)

Note: $a > 0$ and $a \neq 1$

Change of Base: Ex. 1

Find the value of $\log_5 40$

Solution: Change the base and solve

$$\log_a x = \frac{\log_c x}{\log_c a}$$

$$\log_5 40 = \frac{\log_{10} 40}{\log_{10} 5} = 2.292$$

Change of Base: Ex. 2

Find the value of $\log_3 20$

Solution: Change the base and solve

$$\log_a x = \frac{\log_c x}{\log_c a}$$

$$\log_3 20 = \frac{\log_{10} 20}{\log_{10} 3} = 2.727$$

Change of Base: Ex. 3

Find the value of $\log_5 0.1$

Solution: Change the base and solve

$$\log_a x = \frac{\log_c x}{\log_c a}$$

$$\log_5 0.1 = \frac{\log_{10} 0.1}{\log_{10} 5} = -1.43$$

Laws of Logs

There are four important laws to remember when dealing with problems involving logarithms.

1. $\log_b xy = \log_b x + \log_b y$
2. $\log_b \frac{x}{y} = \log_b x - \log_b y$
3. $\log_b x^n = n \log_b x$
4. $\log_b b = 1$ and $\log_b 1 = 0$ for any number b

Why is $\log_b xy = \log_b x + \log_b y$?

Let $\log_b x = p$

$\Rightarrow x = b^p$

Let $\log_b y = q$

$\Rightarrow y = b^q$

Now we can say

$xy = (b^p)(b^q)$

$xy = b^{p+q}$

If we convert this from index form to log form:

$\log_b xy = p + q$

Now replace p and q:

$\log_b xy = \log_b x + \log_b y$

Note: You don't need to know this proof but it explains the rule concisely.

The Exponential Function

The letter "e" represents the exponential function

$e \approx 2.718281828\ldots$

The exponential function is widely used in physics, chemistry and mathematics.

It is commonly used to predict growth e.g. population growth, economy growth, etc.

Natural Logs: Ex. 1

$$e^x = 12$$

Can be rewritten:

$$\log_e 12 = x$$

$$2.485 = x$$

$$x = 2.485$$

$a^b = c$

$\log_a c = b$

Natural Logs: Ex. 2

$$e^{2x+1} = 7$$

Can be rewritten:

$$\log_e 7 = 2x + 1$$

$$1.946 = 2x + 1$$

$$x = 0.473$$

$a^b = c$

$\log_a c = b$

Natural Logs: Ex. 3

$$3e^{x-4} = 19$$

$$e^{x-4} = \frac{19}{3}$$

Can be rewritten:

$$\log_e \left(\frac{19}{3}\right) = x - 4$$

$$1.8458 = x - 4$$

$$x = 5.8458$$

$a^b = c$

$\log_a c = b$

Further Problems: Ex. 1

Solve for x in the following: $3^{x+2} = 48$

Solution: Rewrite as $\log_3 48 = x + 2$

$$3.5237 = x + 2$$

$$x = 1.5237$$

Further Problems: Ex. 2

Solve for x in the following: $7^{3x-1} = 128$

Solution: Rewrite as $\log_7 128 = 3x - 1$

$$2.493 = 3x - 1$$

$$x = 1.1645$$

Laws of Logs

1. $\log_a x + \log_a y = \log_a xy$

Examples:

$$\log_e 5 + \log_e 7 = \log_e 35$$

$$\log_{10} x + \log_{10} 3 = \log_{10} 3x$$

Laws of Logs

2. $\log_a x - \log_a y = \log_a \frac{x}{y}$

Examples:

$$\log_e 15 - \log_e 3 = \log_e 5$$

$$\log_{10} x - \log_{10} 3 = \log_{10} \left(\frac{x}{3}\right)$$

Laws of Logs

3. $\log_a x^p = p \log_a x$

Examples:

$$\log_2 5^3 = 3 \log_2 5$$

$$\log_e x^{0.5} = 0.5 \log_e x$$

Exponential Growth and Decay Problems

Formula: $y = Ae^{kt}$

t = time
k = ratio of growth
A = Initial value
y = value after time t

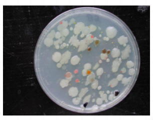
Example 1

The population of a colony of bacteria after t hours is determined by the function:

$$P(t) = 600e^{0.021t}$$

(a) Find the initial population of the colony

(b) How many hours does it take the population to reach 1,000?



Example 1

(a) Find the initial population of the colony

$$P(t) = 600e^{0.021t}$$

t = 0 $\quad P(0) = 600e^{0.021(0)}$

$$P(0) = 600e^{0} = 600(1)$$

Answer: The initial population of the colony is 600

Example 1

(b) How many hours does it take the population to reach 1,000?

$P(t) = 600e^{0.021t}$

$1000 = 600e^{0.021t}$

$1.6666 = e^{0.021t}$ [Divide both sides by 600]

Example 1

$1.6666 = e^{0.021t}$ $\qquad c = a^b$

$\log_e 1.66666 = 0.021t$ $\qquad \log_a c = b$

$0.51082 = 0.021t$

$t = 24.32$ hours

Answer: The colony will increase to a population of 1,000 after 24.32 hours.

Example 2

The value of an investment of €1,000 over t years at 4% per annum compound interest is represented by the following function:

$$y = 1000e^{0.04t}$$

(a) What will the value of the investment be after 10 years?

(b) How long will it take the investment to double?

Example 2

(a) What will the value of the investment be after 10 years?

$y = 1000e^{0.04t}$

$t = 10$ $\qquad y = 1000e^{0.04(10)}$

$y = 1000e^{0.4} = 1000(1.4918)$

$y = €1{,}491.80$

The investment will be worth €1,491.80 after 10 years

Example 2

(b) How long will it take the investment to double?

$y = 1000e^{0.04t}$

$2000 = 1000e^{0.04t}$

$2 = e^{0.04t}$ [Divide both sides by 1,000]

Example 2

$2 = e^{0.04t}$ $\qquad c = a^b$

$\log_e 2 = 0.04t$ $\qquad \log_a c = b$

$0.69315 = 0.04t$

$t = 17.33\ years$

Answer: The investment will double in 17.33 years

Example 3

If radium disintegrates by radioactivity at a rate of 0.00038 of its mass per year:

(a) How much will be left of a 100 gram mass of radium after 1,000 years?

(b) How long will it take 150 grams of radium to reduce to 50 grams?

Example 3(a)

Formula: $y = Ae^{kt}$

t = time = 1,000

k = ratio of growth = -0.00038

A = Initial value = 100

y = value after time t

$$y = 100e^{-0.00038(1000)}$$

Example 3(a)

$$y = 100e^{-0.00038(1000)}$$

$$y = 100e^{-0.38}$$

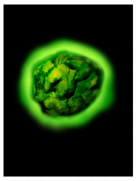
$$y = 100(0.6838)$$

$$y = 68.38 \text{ grams}$$

Answer: 100 grams of radium will reduce to 68.38 grams after 1,000 years

Example 3(b)

(b) How long will it take 150 grams of radium to reduce to 50 grams?



Formula: $y = Ae^{kt}$

t = ?

k = -0.00038

A = 150

y = 50

$\boxed{50 = 150e^{-0.00038t}}$

Example 3(b)

$50 = 150e^{-0.00038t}$

$0.3333 = e^{-0.00038t}$ [Divide both sides by 150]

$\log_e 0.33333 = -0.00038t$

$-1.0986 = -0.00038t$

$t = 2{,}891$ years [Divide both sides by -0.00038]

Example 3(b)

Answer: it will take 2,891 years for 150 grams of radium to reduce to 50 grams.

Example 4

What sum of money, invested for 10 years at 4% per annum compound interest, will grow to €5,000?

Formula: $y = Ae^{kt}$

t = 10

k = 0.04 $\boxed{5{,}000 = Ae^{0.04(10)}}$

A = ?

y = 5,000

Example 4

$5{,}000 = Ae^{0.04(10)}$

$5{,}000 = Ae^{0.4}$

$5{,}000 = A(1.4918)$

$A = €3{,}351.60$

Answer: €3,351.60, invested for 10 years at 4% per annum compound interest, will grow to €5,000.

Example 5

The typical salary of top Premiership soccer players has increased from £10,000 per week to £300,000 per week in the last 20 years. If this rate keeps going, what will be the typical wage of a top premiership player in 10 years?

Example 5

First, find the rate at which wages have been increasing.

Formula: $y = Ae^{kt}$

t = 20 $\boxed{300{,}000 = 10{,}000e^{k(20)}}$

k = ?

A = 10,000

y = 300,000

Example 5

$300{,}000 = 10{,}000e^{20k}$

$30 = e^{20k}$

$\log_e 30 = 20k$

$3.4012 = 20k$

$k = 0.17$

Example 5

Next, find what such a rate will cause the wages to be in 10 years time…

Formula: $y = Ae^{kt}$

t = 10

k = 0.17 $\boxed{y = 300{,}000e^{0.17(10)}}$

A = 300,000

y = ?

Example 5

$$y = 300{,}000e^{1.7}$$

$$y = 300{,}000(5.4739)$$

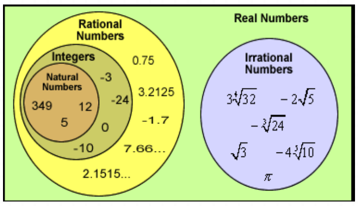
$$y = £1{,}642{,}184.22$$

Conclusion: If the current rate of growth of wages continues, top premiership players will earn around £1.64 *million* per week by the year 2024.

http://www.telegraph.co.uk/sport/football/competitions/premier-league/8265851/How-footballers-wages-have-changed-over-the-years-in-numbers.html

Number Systems

Rational Numbers: Any number that can be expressed as the ratio of two integers i.e. one integer divided by another.



Proof that $\sqrt{2}$ is irrational

How do we know that the square root of 2 is an irrational number?

One proof of the number's irrationality is provided using a techniques called proof by contradiction.

This technique works by assuming that the opposite of the proposition is true and showing that this assumption is false, thereby implying that the proposition must be true

Proof that $\sqrt{2}$ is irrational

1. Assume that $\sqrt{2}$ is a rational number, meaning that there exists an integer a and an integer b in general such that

$$\frac{\mathrm{a}}{\mathrm{b}} = \sqrt{2}$$

2. Then $\sqrt{2}$ can be written as an irreducible fraction $\frac{a}{b}$ such that a and b are co-prime integers.

Proof that $\sqrt{2}$ is irrational

3. If we square both sides:

$$\frac{a^2}{b^2} = 2$$

$$\Rightarrow a^2 = 2b^2$$

4. Therefore a^2 is even because it is equal to $2b^2$ ($2b^2$ is necessarily even because it is 2 times another whole number and even numbers are multiples of 2.)

5. It follows that a must be even (as squares of odd integers are never even).

Proof that $\sqrt{2}$ is irrational

6. Because a is even, there exists an integer k that fulfills: $a = 2k$.

7. Substituting $2k$ from step 6 for a in the second equation of step 3:

$$2b^2 = (2k)^2$$

$$2b^2 = 4k^2$$

$$b^2 = 2k^2$$

Proof that $\sqrt{2}$ is irrational

8. $2k^2$ is divisible by two and therefore even, and $b^2 = 2k^2$ thus it follows that b^2 is also even which means that b is even.

9. By steps 5 and 8 a and b are both even, which contradicts that $\frac{a}{b}$ is irreducible as stated in step 2.

Conclusion: Because there is a contradiction, the assumption (1) that $\sqrt{2}$ is a rational number must be false.

Therefore the opposite is proven: $\sqrt{2}$ is irrational.

Proof by Induction

General Steps:

1. The basis (base case): prove that the statement holds for the first natural number n. Usually, $n = 0$ or $n = 1$.
2. The inductive step: prove that, if the statement holds for some natural number $n = k$, then the statement holds for $n = k + 1$.

Conclusion: The proposition is true for $n = 1$. If the proposition is true for $n = k$, then it will be true for $n = k + 1$. Therefore, by induction, it is true for all $n \in N$

Proof by Induction - Example

Use induction to prove that 8 is a factor of $7^{2n+1} + 1$ for any positive integer n

Method

State proposition:

$P(n)$: $7^{2n+1} + 1$ is divisible by 8 for all positive integers n

Proof by Induction - Example

Prove true for the base case i.e. $n = 1$

$$P(1):\ 7^{2(1)+1} + 1$$

$$= 7^3 + 1$$

$$= 344$$

We know that $344 = 43 \times 8$

Thus it is divisible by 8 and $P(1)$ is true.

Proof by Induction - Example

Assume true for $P(k)$ i.e. when $n = k$

$$P(k):\ 7^{2k+1} + 1$$

We are now assuming that the above is divisible by 8

Prove true for $P(k+1)$ i.e. when $n = k + 1$

$$P(k+1):\ 7^{2(k+1)+1} + 1$$

$$P(k+1):\ 7^{2k+3} + 1$$

Proof by Induction - Example

$$P(k+1):\ 7^{2k+3} + 1$$

$$= (7^2)(7^{2k+1}) + 1$$

$$= (49)(7^{2k+1}) + 1$$

$$= (48 + 1)(7^{2k+1}) + 1$$

$$= (48)(7^{2k+1}) + 1(7^{2k+1}) + 1$$

$$= (48)(7^{2k+1}) + 1(7^{2k+1} + 1)$$

Proof by Induction - Example

$$P(k+1) = \underbrace{(48)(7^{2k+1})}_{\text{This is divisible by } 8 \text{ as } 48 = 6\times 8} + \underbrace{(7^{2k+1}+1)}_{\text{This is divisible by 8 from our assumption}}$$

Thus, $P(k+1)$ is true.

The proposition is true for $n = 1$. If the proposition is true for $n = k$, then it will be true for $n = k + 1$. Therefore, by induction, it is true for all $n \in N$

Proof by Induction – Sample 2014

2014 Sample Paper 1:

Prove by induction that, for any n, the sum of the first n natural numbers is $\frac{n(n+1)}{2}$

Solution:

State proposition:

$$P(n):\ 1+2+3\ldots+n = \frac{n(n+1)}{2}$$

Proof by Induction – Sample 2014

Prove true for $n = 1$

$$1 = \frac{1(1+1)}{2}$$

$$1 = \frac{2}{2}$$

True

Assume true for $P(k)$ i.e. when $n = k$

$$P(k):$$

$$1+2+3\ldots+k = \frac{k(k+1)}{2}$$

We will assume this is true and use this to help complete the next part of our proof.

Proof by Induction – Sample 2014

Prove true for $P(k+1)$ i.e. when $n = k+1$

$$P(k+1):$$

$$1+2+3\ldots+k+(k+1) = \frac{(k+1)(k+1+1)}{2}$$

$$\underbrace{1+2+3\ldots+k}_{\text{From Previous Assumption}}+(k+1) = \frac{(k+1)(k+2)}{2}$$

$$\frac{k(k+1)}{2}+k+1 = \frac{(k+1)(k+2)}{2}$$

Proof by Induction – Sample 2014

$$\frac{k(k+1)}{2}+k+1 = \frac{(k+1)(k+2)}{2}$$

$$\frac{k(k+1)}{2}+\frac{2(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

$$\frac{(k+2)(k+1)}{2} = \frac{(k+1)(k+2)}{2}$$

Thus, $P(k+1)$ is true.

The proposition is true for $n = 1$. If the proposition is true for $n = k$, then it will be true for $n = k + 1$. Therefore, by induction, it is true for all $n \in N$

1. 2012 PM Paper 1 Q4(a)

Prove, by induction, the formula for the sum of the first n terms of a geometric series. That is, prove that, for $r \neq 1$:

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}$$

2. 2011 Paper 1 Q5(c)

Prove by induction that 9 is a factor of $5^{2n+1} + 2^{4n+2}$, for all $n \in N$.

3. 2010 Paper 1 Q5(b)

Use induction to prove that

$$2 + (2 \times 3) + (2 \times 3^2) + \cdots + (2 \times 3^{n-1}) = 3^{n-1};$$

where n is a positive integer.

4. 2011 PM Paper 1 Q1(a)

(i) Explain what it means to say that $\sqrt{3}$ is not a rational number.

(ii) Given a line segment of length one unit, show clearly how to construct a line segment of length $\sqrt{3}$ units, using only a compass and straight edge.

(iii) Solve the equation $x^2 - 2\sqrt{3}x - 9 = 0$, giving your answers in the form $a\sqrt{3}$, where $a \in \mathbb{Q}$.

5. 2011 Paper 1 Q5(b)

(i) Solve the equation:

$$\log_2 x - \log_2(x-1) = 4\log_4 2$$

(ii) Solve the equation:

$$3^{2x+1} - 17(3^x) - 6 = 0$$

Give your answer correct to two decimal places.

6. 2013 Paper 1 Q3

Scientists can estimate the age of certain ancient items by measuring the proportion of carbon−14, relative to the total carbon content in the item. The formula used is

$$Q = e^{\frac{-0.693t}{5730}}$$

where Q is the proportion of carbon−14 remaining and t is the age, in years, of the item.

(a) An item is 2000 years old. Use the formula to find the proportion of carbon−14 in the item.

(b) The proportion of carbon−14 in an item found at Lough Boora, County Offaly, was 0·3402. Estimate, correct to two significant figures, the age of the item.

7. To measure the volume, or "loudness", of a sound, the decibel scale is used. The loudness L, in decibels (dB), of a sound is given by

$$L = 10 \log_{10}\left(\frac{I}{I_0}\right)$$

where I is the intensity of the sound, in watts per square metre (W/m^2), and $I_0 = 10^{-12} W/m^2$. (I_0 is approximately the intensity of the softest sound that can be heard by the human ear.)

(i) The average maximum intensity of sound in a New York subway car is about $3.2 \times 10^{-3} W/m^2$. How loud, in decibels, is the sound level?

(ii) The Occupational Safety and Health Administration in America considers sustained sound levels of 90 dB and above unsafe. What is the intensity of such sounds?

8. In chemistry, the pH of a liquid is a measure of its acidity. We calculate pH as follows:

$$\text{pH} = -\log_{10}(H^+)$$

where H^+ is the hydrogen ion concentration in moles per litre.

(i) The hydrogen ion concentration of human blood is normally about 3.98×10^{-8} moles per litre. Use this information to find the pH of human blood.

(ii) The average pH of seawater is about 8.2. Use this information to find the hydrogen ion concentration of seawater.

(iii) The hydrogen ion concentration of milk is about 1.6×10^{-7} moles per litre. Use this information to find the pH of milk.