Functions 2 – Differentiation Continued & Integration

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Chain Rule

The Chain Rule is a formula for computing the derivative of the composition of two functions.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Chain Rule

Examples of functions that are differentiated using the chain rule:

1. $y = (3x^2 + 17)^4$

2. $y = e^{4x^2 + 6x}$

3. $y = \ln(2x^3 + 7x)$

4. $y = \sqrt{3x^3 + 1}$

5. $y = 3sin(2x^2 + 7)$

6. $y = -4\cos(x^3 - 4x)$

Chain Rule – Ex. 1

Differentiate $y = (2x - 3)^5$ with respect to x.

Let $u = 2x - 3$

$\frac{du}{dx} = 2$

Now: $y = u^5$

$\frac{dy}{du} = 5u^4$

Chain Rule – Ex. 1

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = (5u^4)(2)$$

$$\frac{dy}{dx} = 10u^4 = 10(2x - 3)^4$$

Answer: $\frac{dy}{dx} = 10(2x - 3)^4$

Chain Rule – Ex. 2

Differentiate $y = \ln(cosx^2)$ with respect to x.

Let $u = cosx^2$

$\frac{du}{dx} = -2xsinx^2$

Now: $y = \ln(u)$

$\frac{dy}{du} = \frac{1}{u}$

Special Cases (See Log Tables page 25)

$f(x)$	$f'(x) = \dfrac{d[f(x)]}{dx}$
x^n	nx^{n-1}
$\ln x$	$\dfrac{1}{x}$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$
e^x	e^x
e^{ax}	ae^{ax}

Chain Rule – Ex. 2

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$\frac{dy}{dx} = \left(\frac{1}{u}\right)(-2x\sin x^2)$$

$$= \frac{-2x\sin x^2}{u} = \frac{-2x\sin x^2}{\cos x^2}$$

$$f'(x) = \frac{-2x\sin x^2}{\cos x^2}$$

Steps for applying the Chain Rule

1. Identify u and rewrite y in terms of u.

2. Differentiate u to get $\dfrac{du}{dx}$ and differentiate y to get $\dfrac{dy}{du}$

3. Sub for $\dfrac{dy}{du}$ and $\dfrac{du}{dx}$ in the formula $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

4. Simplify the equation. Replace u with what it was originally.

Chain Rule – A Quicker Way – Ex. 1

Differentiate $y = (2x-3)^5$ with respect to x.

Reduce power by 1

$$\frac{dy}{dx} = 5(2x-3)^4(2)$$

Multiply by the original power

Multiply by the diff of what's inside the brackets

$$\frac{dy}{dx} = 10(2x-3)^4$$

Chain Rule – A Quicker Way – Ex. 2

Differentiate $y = \sqrt{4x-1}$ with respect to x.

Rewrite: $y = (4x-1)^{\frac{1}{2}}$

Reduce power by 1

$$\frac{dy}{dx} = \frac{1}{2}(4x-1)^{\frac{-1}{2}}(4)$$

Multiply by the original power

Multiply by the diff of what's inside the brackets

$$\frac{dy}{dx} = 2(4x-1)^{\frac{-1}{2}} = \frac{2}{\sqrt{4x-1}}$$

Chain Rule – Ex. 3

The revenue received from the sale of electric fans is seasonal, with maximum revenue received in the summer. The revenue received from the sale of fans can be approximated by:

$$R(x) = 100\cos\left(\frac{\pi x}{6}\right) + 120$$

where x is time in months measured from July 1st

Find $R'(x)$ and evaluate $R'(x)$ for (i) August 1st
(ii) May 1st and interpret your answers.

Chain Rule – Ex. 3

$R(x) = 100 \cos\left(\dfrac{\pi x}{6}\right) + 120$

Apply the chain rule:

Let $u = \dfrac{\pi x}{6}$

Now:
$R(u) = 100 \cos(u) + 120$

$\dfrac{dR}{du} = -100 \sin(u)$

$u = \dfrac{\pi x}{6}$

$\dfrac{du}{dx} = \dfrac{\pi}{6}$

We know:

$\dfrac{dR}{dx} = \dfrac{dR}{du} \times \dfrac{du}{dx}$

$\dfrac{dR}{dx} = -100 \sin(u) \times \dfrac{\pi}{6}$

Chain Rule – Ex. 3

$\dfrac{dR}{dx} = -100 \sin(u) \times \dfrac{\pi}{6}$

$\dfrac{dR}{dx} = -\dfrac{50\pi}{3} \sin(u)$

$\dfrac{dR}{dx} = -\dfrac{50\pi}{3} \sin\left(\dfrac{\pi x}{6}\right)$

Chain Rule – Ex. 3

Short Approach to Chain Rule:

$R(x) = 100 \cos\left(\dfrac{\pi x}{6}\right) + 120$

$R'(x) = \left(\dfrac{\pi}{6}\right) 100 \left(-\sin\left(\dfrac{\pi x}{6}\right)\right)$

$R'(x) = -\dfrac{50\pi}{3} \sin\left(\dfrac{\pi x}{6}\right)$

Note: $R'(x)$ means the same as $\dfrac{dR}{dx}$.

Chain Rule – Ex. 3

$R'(x) = -\dfrac{50\pi}{3} \sin\left(\dfrac{\pi x}{6}\right)$

(i) $R'(x)$ for August 1st

$x = 1$ as August 1st is one month after July 1st

$R'(1) = -\dfrac{50\pi}{3} \sin\left(\dfrac{\pi(1)}{6}\right)$

$R'(1) = -26.18$

Rate of change is negative – this means that the revenue received from the sale of fans is decreasing in August.

Chain Rule – Ex. 3

$R'(x) = -\dfrac{50\pi}{3} \sin\left(\dfrac{\pi x}{6}\right)$

(ii) $R'(x)$ for May 1st

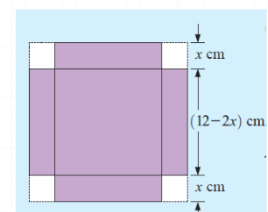$x = 10$ as May 1st is ten months after July 1st

$R'(10) = -\dfrac{50\pi}{3} \sin\left(\dfrac{\pi(10)}{6}\right)$

$R'(10) = 45.345$

Rate of change is positive – this means that the revenue received from the sale of fans is increasing in May.
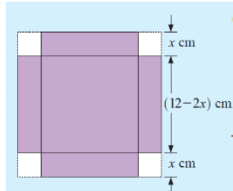

Optimisation – Ex. 3

Square corners are cut from a piece of 12 cm by 12 cm tinplate which is then bent to form an open dish. What size squares should be removed to maximise the capacity of the dish?

Optimisation – Ex. 3

Capacity = length × width × depth

$$C = (12 - 2x)(12 - 2x)(x)$$

$$C = 144x - 48x^2 + 4x^3$$

1. Differentiate Capacity to find $\frac{dC}{dx}$
2. Let $\frac{dC}{dx} = 0$
3. Find value for x

Optimisation – Ex. 3

$C = 144x - 48x^2 + 4x^3$

$$\frac{dC}{dx} = 144 - 96x + 12x^2$$

Let $\frac{dC}{dx} = 0$

$$144 - 96x + 12x^2 = 0$$

$$12x^2 - 96x + 144 = 0$$

[Divide across by 12 to simplify]

$$x^2 - 8x + 12 = 0$$

$$(x - 2)(x - 6) = 0$$

$$x = 2 \quad \text{or} \quad x = 6$$
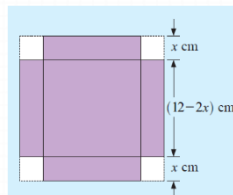

Optimisation – Ex. 3

We take $x = 2$ because $x = 6$ would not be possible (refer to the diagram)

Max capacity at $x = 2$

$$C = 144x - 48x^2 + 4x^3$$

$$C = 144(2) - 48(2)^2 + 4(2)^3$$

Maximum Capacity $= 128 cm^3$

Calculus Problem in the Real World

An oil rig springs a leak in calm seas and the oil spreads in a circular patch around the rig. If the radius of the oil patch increases at a rate of 30 metres/hour, how fast is the area of the patch increasing when the patch has a radius of 100 metres.

Also consider what will happen as the radius increases?

Calculus Problem in the Real World

Two variables change simultaneously: the radius of the circle and its area.

The first key thing we must do is establish some kind of relationship between these two variables and we can draw on our knowledge of geometry to do this:

$$A = \pi r^2$$

We are asked to identify how fast the **area** of the patch is increasing i.e. $\frac{dA}{dt}$.

We are told the **radius** increases at a rate of 30 m/hr.

$$\frac{dr}{dt} = 30$$

Calculus Problem in the Real World

Now we will try to write $\frac{dA}{dt}$ in terms of $\frac{dr}{dt}$.

$$\frac{dA}{dt} = \frac{dr}{dt} \times ?$$

$$\frac{dA}{dt} = \frac{dr}{dt} \times \frac{dA}{dr}$$

We know $A = \pi r^2$
If we differentiate this:

$$\frac{dA}{dr} = 2\pi r$$

Now we can solve

$$\frac{dA}{dt} = \frac{dr}{dt} \times \frac{dA}{dr}$$

$$\frac{dA}{dt} = (30)(2\pi r)$$

Calculus Problem in the Real World

$$\frac{dA}{dt} = (30)(2\pi r)$$

$$\frac{dA}{dt} = 60\pi r$$

Conclusion: the area of the patch is increasing at a rate of $60\pi r$

We are asked for the rate of increase in area when the radius is 100 metres

$$\frac{dA}{dt} = 60\pi r = 60\pi(100)$$

$$= 18,850 \ m^2/hr$$

When the radius of the oil patch is 100 m, the oil patch is increasing at a rate of $18,850 \ m^2/hr$

What is Integration?

Integration is the reverse process of differentiation.

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c$$

Add one to the power   Divide by new power

"c" represents the constant that may be present (to be explained later).

Applications of Integration

Integration is used to calculate the following:

- Moments of Inertia.
- Volume of a solid revolution.
- Electric Charges.
- Force by a liquid pressure.
- Area under a curve.
- Area between two curves.
- Work by a variable force.
- displacement, Velocity, Acceleration.
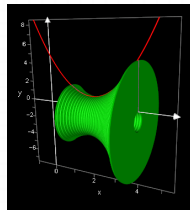
Integration and Differentiation
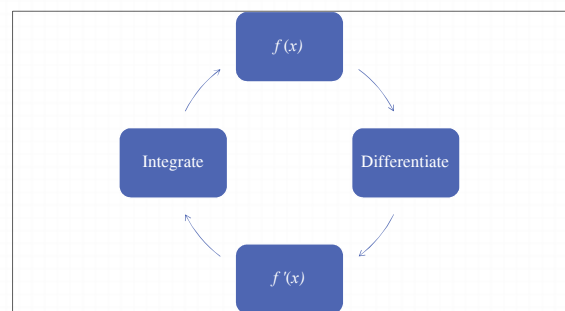
Integration and Differentiation

If we start with the function $f(x) = 3x^2 + 4x - 10$

Differentiate $f(x)$

$$\frac{df}{dx} = 6x + 4$$

Integrate to reverse this process:

$$\int df = \int 6x + 4 \ dx$$

$$f(x) = \frac{6x^2}{2} + 4x + c$$

$$f(x) = 3x^2 + 4x + c$$

Integration and Differentiation

We started with
$$f(x) = 3x^2 + 4x - 10$$

Integration reversed the process of differentiation and we ended up with
$$f(x) = 3x^2 + 4x + c$$

The constant (–10) was eliminated during differentiation thus the reason for including "c" when integrating.

Rules for Integration

$$\int k \, dx = kx + c \qquad [k \text{ is a constant}]$$

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \qquad [n \neq -1]$$

$$\int k \, f(x) = k \int f(x) \qquad [k \text{ is a constant}]$$

$$\int [f(x) \pm g(x)] dx = \int f(x) \, dx \pm \int g(x) \, dx$$

COMMON INTEGRALS

$$\int k \, dx = kx + C$$

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, n \neq -1$$

$$\int x^{-1} \, dx = \int \frac{1}{x} \, dx = \ln|x| + C$$

$$\int \frac{1}{ax+b} \, dx = \frac{1}{a} \ln|ax+b| + C$$

$$\int \ln(x) \, dx = x\ln(x) - x + C$$

$$\int e^x \, dx = e^x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \csc x \cot x \, dx = -\csc x + C$$

$$\int \csc^2 x \, dx = -\cot x + C$$

$$\int \tan x \, dx = \ln|\sec x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

$$\int \frac{1}{a^2+u^2} \, dx = \frac{1}{a} \tan^{-1}\left(\frac{u}{a}\right) + C$$

$$\int \frac{1}{\sqrt{a^2-u^2}} \, dx = \sin^{-1}\left(\frac{u}{a}\right) + C$$

Basic Examples

$$\int x^2 + 2\cos x \, dx$$

$$= \frac{x^3}{3} + 2\sin x + c$$

$$\int 4e^x + \frac{1}{x} - 20$$

$$= 4e^x + lnx - 20x + c$$

Remember...

When differentiating the product of two functions we used the product rule:

$$y = uv$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

When differentiating the quotient of two functions we used the quotient rule:

$$y = \frac{u}{v}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

Remember...

When finding the derivative of the composition of two functions, we use the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Applying Integration to Dynamics

When we differentiate w.r.t. time a function representing the displacement of an object, we obtain an equation for the velocity of that object. If we differentiate again, we get an equation for the acceleration of that object.

Displacement → Velocity → Acceleration

As integration is the opposite of differentiation, the same process can be applied through integration but in reverse:

Acceleration → Velocity → Displacement

Applying Integration to Dynamics

Acceleration: $a(t)$

Velocity: $v(t)$

Displacement: $s(t)$

$$\int a(t)dt = v(t)$$

$$\int v(t)dt = s(t)$$

Dynamics – Ex. 1

Q. A car has an acceleration:

$$a(t) = 2 + 6t$$

The car starts from rest at time $t = 0$ from position $s = 10$. Find its position and velocity at all times t.

Find velocity first:

$$v(t) = \int a(t)\,dt$$

$$v(t) = \int (2 + 6t)dt$$

$$v(t) = 2t + 3t^2 + c$$

Dynamics – Ex. 1

We know that velocity was zero at $t = 0$. This info will help us find out what c is:

$$v(t) = 2t + 3t^2 + c$$

$$v(0) = 2(0) + 3(0)^2 + c$$

$$v(0) = c$$

Velocity was zero at $t = 0$

$$v(0) = 0$$

Thus, $c = 0$

$$v(t) = 2t + 3t^2$$

This is the velocity of the car at all times t.

Next, find the displacement of the car $s(t)$

$$s(t) = \int v(t)\,dt$$

$$s(t) = \int (2t + 3t^2)dt$$

Dynamics – Ex. 1

$$s(t) = \int 2t + 3t^2$$

$$s(t) = t^2 + t^3 + c$$

We know that the car started from rest at time $t = 0$ from position $s = 10$. This info will help us find c.

$$s(0) = (0)^2 + (0)^3 + c$$

$$s(0) = c$$

We know $s(0) = 10$

Thus, $c = 10$

$$s(t) = t^2 + t^3 + 10$$

This is the position of the car at all times t.

Dynamics – Ex. 2

Q. A boat has a velocity:

$$v(t) = 3t^2 + 4t$$

The car starts from rest at time $t = 0$ from position $s = 4$. Find its position at all times t.

$$s(t) = \int v(t)dt$$

$$s(t) = \int (3t^2 + 4t)dt$$

$$s(t) = \frac{3t^3}{3} + \frac{4t^2}{2} + c$$

$$s(t) = t^3 + 2t^2 + c$$

Dynamics – Ex. 2

$$s(t) = t^3 + 2t^2 + c$$

Remember: the car starts from rest at time $t = 0$ from position $s = 4$.

Thus:

$$s(0) = 4$$

$$s(t) = t^3 + 2t^2 + c$$

$$s(0) = (0)^3 + 2(0)^2 + c$$

$$s(0) = c$$

We know: $s(0) = 4$

$$c = 4$$

$$s(t) = t^3 + 2t^2 + 4$$

Dynamics – Ex. 3

Q. When an object is dropped, its acceleration (ignoring air resistance) is constant at $9.8 \ m/s^2$

$$a(t) = -9.8$$

The negative sign is used because the object is falling.

Suppose an object is thrown from the 381 metre high rooftop of the Empire State Building in New York. If the initial velocity of the object is $-6 \ m/s$, find out how long it takes the object to hit the ground and the velocity at which the object is travelling just before it hits the ground.

Dynamics – Ex. 3

$$a(t) = -9.8$$

To find the velocity of the object we must integrate the acceleration:

$$v(t) = \int a(t) \, dt$$

$$v(t) = \int (-9.8) \, dt$$

$$v(t) = -9.8t + c$$

We know the initial velocity was $-6 \ m/s$ so we can find "c" using this information:

$$v(0) = -9.8(0) + c$$

$$v(0) = c$$

But we know:
$$v(0) = -6$$
Thus
$$c = -6$$

$$v(t) = -9.8t - 6$$

Dynamics – Ex. 3

$$v(t) = -9.8t - 6$$

To find the distance of the object from the ground, we must integrate the velocity:

$$s(t) = \int v(t) \, dt$$

$$s(t) = \int (-9.8t - 6) \, dt$$

$$s(t) = -4.9t^2 - 6t + c$$

We know the initial distance from the ground was 381 metres so we can find "c" using this information:

$$s(0) = -9.8(0)^2 - 6(0) + c$$

$$s(0) = c$$

$$s(0) = 381$$
Thus
$$c = 381$$

$$s(t) = -4.9t^2 - 6t + 381$$

Dynamics – Ex. 3

When the object hits the ground, its distance from the ground will be zero:

$$s(t) = -4.9t^2 - 6t + 381$$

$$-4.9t^2 - 6t + 381 = 0$$

Solve for t using quadratic formula:

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-(-6) \pm \sqrt{(-6)^2 - 4(-4.9)(381)}}{2(-4.9)}$$

$$= \frac{6 \pm \sqrt{7503.6}}{-9.8}$$

$$t = -9.45 \ or \ 8.23$$

We take our value for time as 8.23 seconds, because a negative value would indicate going backwards in time.

The object takes 8.23 seconds to reach the ground.

Dynamics – Ex. 3

To find the velocity of the object just before it hits the ground, substitute 8.23 instead of t in the equation for the velocity of the object as it falls:

$$v(t) = -9.8t - 6$$

$$v(8.23) = -9.8(8.23) - 6$$

$$v(8.23) = -86.654$$

This indicates that the object was travelling at a velocity of $86.654 \ m/s$ just as it hit the ground.
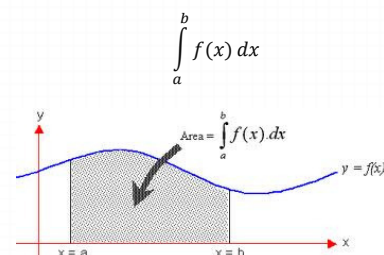
This is equivalent to a velocity of $311.95 \ km/h$

The minus indicates that it was moving in a downward direction i.e. towards the ground.

Note: air resistance would alter this velocity considerably.

The Definite Integral

To determine the area under a curve $f(x)$ between $x = a$ and $x = b$ we use the definite integral:

$$\int_a^b f(x) \, dx$$

Definite Integral – Ex. 1

Evaluate:

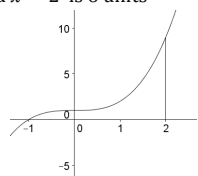$$\int_0^2 x^3 + 1$$

$$= \left[\frac{x^4}{4} + x\right]_0^2$$

Sub in the limits 2 and 0

$$= \left(\frac{(2)^4}{4} + 2\right) - \left(\frac{(0)^4}{4} + 0\right)$$

$$= 6 - 0$$

$$= 6$$

The area under the curve $f(x) = x^3 + 1$ between $x = 0$ and $x = 2$ is 6 units^2

Definite Integral – Ex. 1

Note: when the function was integrated, the "c" that is normally added on was left out.

The reason for this is that when calculating the definite integral, "c" will be cancelled out every time:

$$\int_0^2 (x^3 + 1)\,dx$$

$$= \left[\frac{x^4}{4} + x + c\right]_0^2$$

Sub in the limits 2 and 0

$$= \left(\frac{(2)^4}{4} + 2 + c\right) - \left(\frac{(0)^4}{4} + 0 + c\right)$$

$$= 6 + c - 0 - c = 6$$

"c" always cancels so we just leave it out at the start.

Definite Integral – Ex. 2

Find the area between $f(x) = \sin x$ and the x axis between $x = 0$ and $x = \pi$

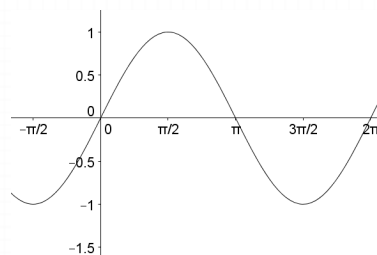Solution:

$$\int_0^\pi \sin x\, dx$$

$$= [-\cos x]_0^\pi$$

$$= (-\cos \pi) - (-\cos 0)$$

$$= (1) - (-1)$$

$$= 2$$

The area between $f(x) = \sin x$ and the x axis between $x = 0$ and $x = \pi$ is 2 units^2

Definite Integral – Ex. 2

Definite Integral – Ex. 3

Find the area between $f(x) = \sin x$ and the x axis between $x = \pi$ and $x = 2\pi$

Solution:

$$\int_\pi^{2\pi} \sin x\, dx$$

$$= [-\cos x]_\pi^{2\pi}$$

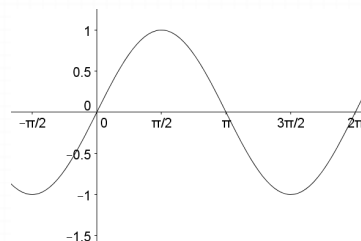$$= (-\cos 2\pi) - (-\cos \pi)$$

$$= (-1) - (1)$$

$$= -2$$

Because it is an area that we are calculating, we ignore the minus.

The area between $f(x) = \sin x$ and the x axis between $x = \pi$ and $x = 2\pi$ is 2 units^2

Definite Integral – Ex. 3

When the area is below the x axis, the calculation of the area between the curve and the x axis will produce a negative value as it did in this example.

Definite Integral – Ex. 4

Find the area between $f(x) = \sin x$ and the x axis between $x = 0$ and $x = 2\pi$

Solution:

$$\int_0^{2\pi} \sin x\, dx$$

$$= [-\cos x]_0^{2\pi}$$

$= (-\cos 2\pi) - (-\cos 0)$

$$= (-1) - (-1)$$

$$= 0$$

This suggest that there is no area between $f(x) = \sin x$ and the x axis between $x = 0$ and $x = 2\pi$

But if we look at the graph of $f(x) = \sin x$ we can see that this is obviously not true…

Definite Integral – Ex. 4

The positive area between $x = 0$ and $x = \pi$ was cancelled out by the negative area between $x = \pi$ and $x = 2\pi$. As such, the area between the curve and the x axis above the x axis needs to be calculated separately to the area between the curve and the x axis below the x axis, then combined to find the full area.
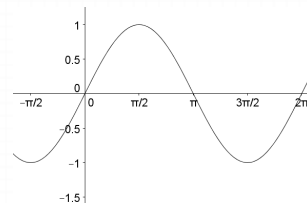
Definite Integral – Ex. 4

In Example 2, we found:

$$\int_0^{\pi} \sin x\, dx = 2 \text{ units}^2$$

In Example 3, we found:

$$\int_{\pi}^{2\pi} \sin x\, dx = 2 \text{ units}^2$$

This means that the area between $f(x) = \sin x$ and the x axis between $x = 0$ and $x = 2\pi$ is:

$$2 + 2 = 4 \text{ units}^2$$

Definite Integral – Ex. 5

Find the area between $f(x) = x^2 + 4x$ and the x axis between $x = -4$ and $x = 3$

Solution:

We must check if the curve crosses the x axis and how that might affect our calculations.

$$f(x) = x^2 + 4x$$

Crosses x axis when $y = 0$

$$0 = x^2 + 4x$$

$$x(x + 4) = 0$$

$$x = 0 \quad or \quad x = -4$$

Crosses x axis at 0 and –4

Definite Integral – Ex. 5
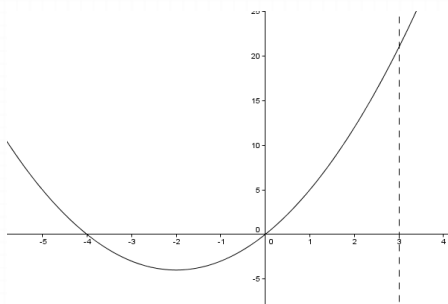
Definite Integral – Ex. 5

As such, we need to find the area below the x axis and the area above the x axis separately.

We'll start with the area below the x axis:

$$\int_{-4}^{0} (x^2 + 4x)\, dx$$

$$= \left[\frac{x^3}{3} + 2x^2\right]_{-4}^{0}$$

$$= \left(\frac{(0)^3}{3} + 2(0)^2\right)$$

$$- \left(\frac{(-4)^3}{3} + 2(-4)^2\right)$$

$$= 0 - \left(-\frac{64}{3} + 32\right)$$

$$= -\frac{32}{3}$$

The area is $\frac{32}{3}$ units^2

Definite Integral – Ex. 5

Next we need to find the area above the x axis:

$$\int_0^3 (x^2+4x)\,dx$$

$$= \left[\frac{x^3}{3} + 2x^2\right]_0^3$$

$$= \left(\frac{(3)^3}{3} + 2(3)^2\right)$$
$$- \left(\frac{(0)^3}{3} + 2(0)^2\right)$$

$$= (9 + 18) - (0)$$

$$= 27 \text{ units}^2$$

The area above the x axis is 27 units²

Definite Integral – Ex. 5

To find the area between $f(x) = x^2 + 4x$ and the x axis between $x = -4$ and $x = 3$

We add the two areas we found:

$$Total\ Area = \frac{32}{3} + 27 = 37\frac{2}{3}$$

Answer: Area is $37\frac{2}{3}$ units²

Definite Integrals – Ex. 6

Q. Calculate the following:

$$\int_0^1 \frac{4x^3}{x^4 + 1}\,dx$$

Integrating the quotient of two functions requires us to use substitution.

Let $u = x^4 + 1$

$$\frac{du}{dx} = 4x^3$$

$$\frac{du}{4x^3} = dx$$

Now, we must also change the limits:

$x = 1 \qquad u = (1)^4 + 1 = 2$

$x = 0 \qquad u = (0)^4 + 1 = 1$

Definite Integrals – Ex. 6

$$\int_0^1 \frac{4x^3}{x^4 + 1}\,dx$$

$$= \int_1^2 \frac{4x^3}{u}\,\frac{du}{4x^3}$$

$$= \int_1^2 \frac{1}{u}\,du$$

$$= [ln|u|]_1^2$$

$$= ln|2| - ln|1|$$

$$= 0.693 - 0$$

$$= 0.693$$

Explaining Integration

If we know the rate at which a quantity is changing, we can find the total change over a period of time by integrating.

For example, if we know the equation for the velocity of an object, which is the rate of change in displacement over time, then integrating it will give us the displacement over a certain period of time.

Another example: If we know the increase in length per year of the horn of a bighorn ram then integrating the function that represents this growth will allow us to calculate the total growth of the horn over a number of years.

Rams

The average annual increase in the horn length (in cm) of bighorn rams can be approximated by
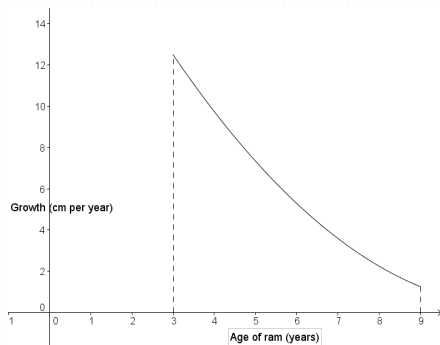
$$y = 0.1762x^2 - 3.986x + 22.68$$

Where x is the ram's age in years for $3 \leq x \leq 9$

Integrate this function to find the total increase in the length of a ram's horn during this time.

Rams

Rams

$$y = 0.1762x^2 - 3.986x + 22.68$$

Total growth between the ages of 3 and 9:

$$\int_{3}^{9} 0.1762x^2 - 3.986x + 22.68$$

$$= \frac{0.1762x^3}{3} - \frac{3.986x^2}{2} + 22.68x \Big|_{3}^{9}$$

$$= 0.0587x^3 - 1.993x^2 + 22.68x \Big|_{3}^{9}$$

Rams

$$= 0.0587x^3 - 1.993x^2 + 22.68x \Big|_{3}^{9}$$

$$= [0.0587(9)^3 - 1.993(9)^2 + 22.68(9)]$$
$$- [0.0587(3)^3 - 1.993(3)^2 + 22.68(3)]$$

$$= 85.4793 - 51.6879$$

$$= 33.79$$

The total increase in the length of a ram's horn between the ages of 3 and 9 was 33.79 cm

Area Problems – Ex. 1

Q. Find the area enclosed by the curve $f(x) = x^2 - x$ and the x axis.
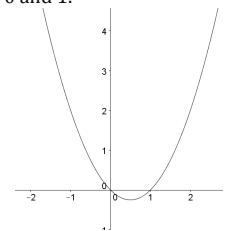
Solution:

First, find where the curve cuts the x axis:

$$x^2 - x = 0$$

$$x(x - 1) = 0$$

$x = 0 \quad or \quad x = 1$

$f(x) = x^2 - x$ cuts the x axis at 0 and 1.

Area Problems – Ex. 1

This gives us the limits of our definite integral. To find the area between the curve and the x axis, we must calculate the following:

$$\int_{0}^{1} (x^2 - x)\, dx$$

$$\int_{0}^{1} (x^2 - x)\, dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} \right]_{0}^{1}$$

$$= \left(\frac{(1)^3}{3} - \frac{(1)^2}{2} \right) - \left(\frac{(0)^3}{3} - \frac{(0)^2}{2} \right)$$

$$= -\frac{1}{6}$$

Area Problems – Ex. 1

$$\int_{0}^{1} (x^2 - x)\, dx = -\frac{1}{6}$$

This means that the area enclosed by the curve $f(x) = x^2 - x$ and the x axis is $\frac{1}{6}\ units^2$

Area Problems – Ex. 2

Q. Find the area enclosed by the curve $f(x) = x^3 - 4x$ and the x axis.
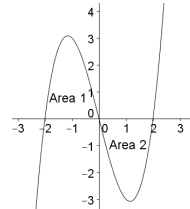
Solution:

First, find where the curve cuts the x axis:

$$x^3 - 4x = 0$$

$$x(x^2 - 4) = 0$$

$x = 0$	$x^2 = 4$
	$x = \pm 2$

Curve cuts the x axis at –2, 0 and 2

Area Problems – Ex. 2

We must find Area 1 and Area 2 separately.

Area 1:

$$\int_{-2}^{0} (x^3 - 4x)\, dx$$

$$\left[\frac{x^4}{4} - 2x^2\right]_{-2}^{0}$$

$$= \left(\frac{(0)^4}{4} - 2(0)^2\right)$$
$$- \left(\frac{(-2)^4}{4} - 2(-2)^2\right)$$

$$= 0 - (-4)$$

$$= 4\ units^2$$

Area Problems – Ex. 2

Area 2:

$$\int_{0}^{2} (x^3 - 4x)\, dx$$

$$\left[\frac{x^4}{4} - 2x^2\right]_{0}^{2}$$

$$= \left(\frac{(2)^4}{4} - 2(2)^2\right)$$
$$- \left(\frac{(0)^4}{4} - 2(0)^2\right)$$

$$= -4$$

Area 1: $4\ units^2$
Area 2: $4\ units^2$

Total Area = $8\ units^2$

The area enclosed by the curve $f(x) = x^3 - 4x$ and the x axis is $8\ units^2$

Improper Integrals

Improper Integrals are integrals involving infinity.

Example: The current flowing into a capacitor at time t is e^{-t} amps.

Find the total charge (as $t \to \infty$) which builds up on the capacitor which is initially uncharged.

Improper Integrals – Ex. 1

Solution:
Find the total charge (q) by integrating the function representing current ($i(t)$)

$$q = \int_{0}^{\infty} e^{-t}\, dt$$

$$q = \left[-e^{-t}\right]_{0}^{\infty}$$

$$q = (-e^{-\infty}) - (-e^0)$$
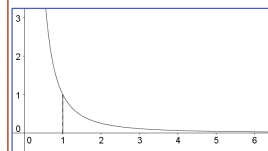
$$q = \left(-\frac{1}{e^{\infty}}\right) - (-1)$$

$$q = \left(-\frac{1}{\infty}\right) + 1$$

$$q = 0 + 1 = 1$$

The total charge is 1 coulomb.

Improper Integrals – Ex. 2

Q. Find the area enclosed between $y = \frac{1}{x^2}$ and the x axis for $x \geq 1$

Solution: Evaluate

$$\int_{1}^{\infty} \frac{1}{x^2}\, dx$$

$$= \int_{1}^{\infty} x^{-2}\, dx$$

$$= \left[\frac{x^{-1}}{-1}\right]_{1}^{\infty}$$

Improper Integrals – Ex. 2

$$= \left[\frac{x^{-1}}{-1}\right]_1^\infty$$

$$= \left[\frac{-1}{x}\right]_1^\infty$$

$$= \left(-\frac{1}{\infty}\right) - \left(-\frac{1}{1}\right)$$

$$= 0 + 1 = 1$$

The area enclosed between $y = \frac{1}{x^2}$ and the x axis for $x \geq 1$ is $1\ unit^2$

Finding the Area between two curves

To find the area between two curves $g(x)$ and $f(x)$ with points of intersection at $x = a$ and $x = b$, we use the following approach:

$$\int_a^b [g(x) - f(x)]\, dx$$

Area between two curves – Ex. 1

Find the area of the region bounded by the curve $y = -x^2 + 5x - 4$ and the line $y = x - 1$

Solution:

1. Sketch the curve and the line then find the points of intersection of the curve and the line.

2. Use integration methods to find the area between the line and curve:

$$\int_a^b [g(x) - f(x)]\, dx$$

Area between two curves – Ex. 1

$$y = -x^2 + 5x - 4$$
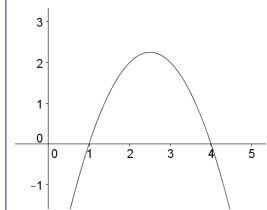
x intercepts:

$$-x^2 + 5x - 4 = 0$$

$$x^2 - 5x + 4 = 0$$

$$(x - 4)(x - 1) = 0$$

$$x = 4 \qquad x = 1$$

Curve cuts x axis at 1 and 4.

Area between two curves – Ex. 1

Find the points of intersection of the line and the curve using simultaneous equations:
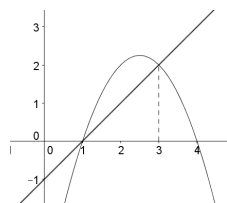
$$y = -x^2 + 5x - 4$$
$$y = x - 1$$

$$-x^2 + 5x - 4 = x - 1$$

$$x^2 - 4x + 3 = 0$$

$$(x - 3)(x - 1) = 0$$

$$x = 3 \qquad x = 1$$

The line and the curve intersect at $x = 3$ and at $x = 1$

Area between two curves – Ex. 1

We now know the equation for the two curves:

$$y = -x^2 + 5x - 4$$
$$y = x - 1$$

And the x values at which they intersect: 1 and 3.

So we can apply:

$$\int_a^b [g(x) - f(x)]\, dx$$

$$\int_1^3 [(-x^2 + 5x - 4) - (x - 1)]dx$$

$$\int_1^3 (-x^2 + 4x - 3)dx$$

$$= \left[-\frac{x^3}{3} + 2x^2 - 3x\right]_1^3$$

Area between two curves – Ex. 1

$$= \left[-\frac{x^3}{3} + 2x^2 - 3x\right]_1^3$$

$$= \left(-\frac{(3)^3}{3} + 2(3)^2 - 3(3)\right)$$
$$- \left(-\frac{(1)^3}{3} + 2(1)^2 - 3(1)\right)$$

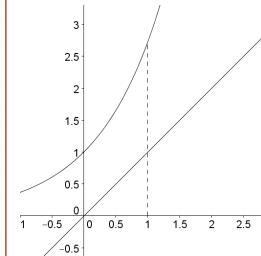$$= (-9 + 18 - 9) - \left(-\frac{1}{3} - 1\right)$$

$$= 0 - \left(-\frac{4}{3}\right)$$

$$= \frac{4}{3} \; units^2$$

The area of the region bounded by the curve $y = -x^2 + 5x - 4$ and the line $y = x - 1$ is $\frac{4}{3} \; units^2$

Area between two curves – Ex. 2

Q. Find the area between $y = e^x$ and $y = x$ for $0 \le x \le 1$

$$\int_a^b [g(x) - f(x)] \, dx$$

$$\int_0^1 (e^x - x) \, dx$$

$$= \left[e^x - \frac{x^2}{2}\right]_0^1$$

Area between two curves – Ex. 2

$$= \left[e^x - \frac{x^2}{2}\right]_0^1$$

$$= \left(e^1 - \frac{(1)^2}{2}\right) - \left(e^0 - \frac{(0)^2}{2}\right)$$

$$= e - \frac{1}{2} - 1$$

$$= 1.218 \; units^2$$

The area between $y = e^x$ and $y = x$ for $0 \le x \le 1$ is
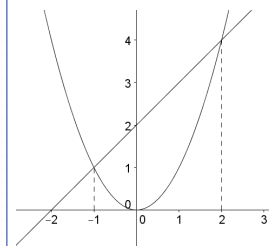
$$1.218 \; units^2$$

Area between two curves – Ex. 3

Find the area between $y = x^2$ and $y = x + 2$

Solution:

Sketch the curves and find where they intersect.

Area between two curves – Ex. 3

Use simultaneous equations to find where they intersect:

$$y = x^2$$
$$y = x + 2$$

$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0$$

$$x = 2 \qquad x = -1$$

The curve and the line intersect at $x = 2$ and $x = -1$

Now we can apply:

$$\int_a^b [g(x) - f(x)] \, dx$$

$$\int_{-1}^2 [(x^2) - (x + 2)] \, dx$$

Area between two curves – Ex. 3

$$\int_{-1}^2 [(x^2) - (x + 2)] \, dx$$

$$\int_{-1}^2 (x^2 - x - 2) \, dx$$

$$= \left[\frac{x^3}{3} - \frac{x^2}{2} - 2x\right]_{-1}^2$$

$$= \left(\frac{(2)^3}{3} - \frac{(2)^2}{2} - 2(2)\right)$$
$$- \left(\frac{(-1)^3}{3} - \frac{(-1)^2}{2} - 2(-1)\right)$$

$$= \left(\frac{8}{3} - 6\right) - \left(-\frac{5}{6} + 2\right)$$

$$= -4\frac{1}{2}$$

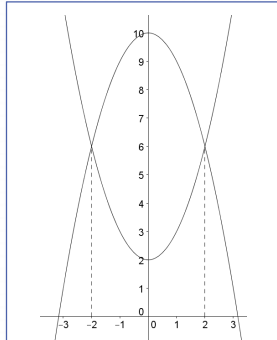$$Area = 4\frac{1}{2} \; units^2$$

Area between two curves – Ex. 4

Find the area enclosed by the curves $y = x^2 + 2$ and $y = 10 - x^2$

Solution:
Find the points at which the curves intersect.

Area between two curves – Ex. 4

Point of Intersection:

$$y = x^2 + 2$$
$$y = 10 - x^2$$

$$x^2 + 2 = 10 - x^2$$

$$2x^2 = 8$$

$$x = \pm 2$$

Curves intersect at $x = 2$ and at $x = -2$

Now we can find the area between the curves using:

$$\int_a^b [g(x) - f(x)]\, dx$$

$$\int_{-2}^{2} [(x^2 + 2) - (10 - x^2)]\, dx$$

Area between two curves – Ex. 4

$$\int_{-2}^{2} [(x^2 + 2) - (10 - x^2)]\, dx$$

$$= \int_{-2}^{2} (2x^2 - 8)\, dx$$

$$= \left[\frac{2x^3}{3} - 8x\right]_{-2}^{2}$$

$$= \left(\frac{2(2)^3}{3} - 8(2)\right)$$

$$- \left(\frac{2(-2)^3}{3} - 8(-2)\right)$$

$$= \left(\frac{16}{3} - 16\right) - \left(-\frac{16}{3} + 16\right)$$

$$= -21\frac{1}{3}$$

$$\text{Area} = 21\frac{1}{3} units^2$$

Q1

The concentration in parts per million (ppm) of carbon dioxide in the atmosphere can be approximated by

$$C(x) = 353(1.0061)^{x-1990}$$

where x is the year. Find and interpret $C'(2014)$.

Q2

The growth of a population of rare South American beetles is given by the function

$$G(t) = \frac{10{,}000}{1 + 49e^{-0.1t}}$$

where t is time in months since the initial count.

    (a) Find the initial count and maximum population of the beetles.

    (b) Find the population and rate of growth of the beetles after (i) 6 months (ii) 3 years.

Q3

Tim is swinging slowly on a garden swing.

The distance, s metres, of Tim below the point, O, from where the swing is suspended can be modelled by the equation

$$s = 2.1 + 0.2\cos\left(\frac{3\pi t}{2}\right)$$

where t is the time in seconds after Tim is first at his lowest point.

    (a) Find the distance below O predicted by the model when $t = \frac{2}{3}$

    (b) (i) Show that the model predicts that Tim is at his highest point when $t = 2$
        (ii) Find when the model predicts the next highest point.

    (c) Find an expression for $\dfrac{ds}{dt}$

Q4 (2012 LC Pilot Paper)

A spherical snowball is melting at a rate proportional to its surface area. That is, the rate at which its volume is decreasing at any instant is proportional to its surface area at that instant.

(i)   Prove that the radius of the snowball is decreasing at a constant rate.

(ii)   If the snowball loses half of its volume in an hour, how long more will it take for it to melt completely? Give your answer correct to the nearest minute.

Q5

(a) Find the area enclosed by the curve $y = 4 - x^2$ and the line $y = x + 2$

(b) Find the area enclosed by the curve $f(x) = 3x^2 - 4x + 1$ and the x axis.

Q6

A population of E. coli bacteria will grow at a rate given by

$$w'(t) = (3t + 4)^{\frac{1}{3}}$$

where $w(t)$ is the weight in milligrams after t hours. Find the total change in weight of the population in the first 5 hours i.e. from $t = 0$ to $t = 5$.