Algebra 2 - Complex Numbers

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Why do we need Complex Numbers?

Some equations have no real solutions.

For instance, the quadratic equation:

$$x^2 + 1 = 0$$

has no real solution because there is no real number x that can be squared to produce -1

Why do we need Complex Numbers?

To overcome this deficiency, mathematicians created an expanded system of numbers using the imaginary unit i, defined as

$$i = \sqrt{-1}$$

Where: $i^2 = -1$

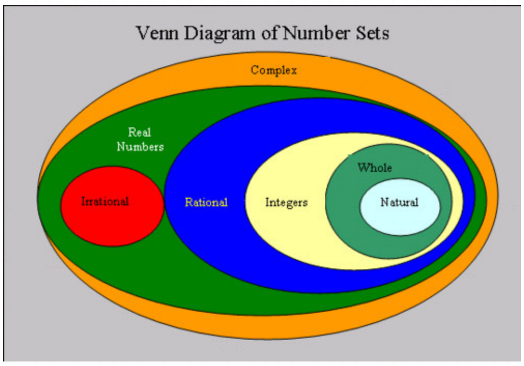
Why do we need Complex Numbers?

By adding real numbers to real multiples of this imaginary unit, we obtain the set of complex numbers.

Each complex number can be written in the standard form:

$$a + bi$$

Complex Numbers



Where are Complex Numbers used?

- Electronic Engineering
- Aircraft Design
- Medicine
- Movie/Computer game graphics
  (http://plus.maths.org/content/maths-goes-movies)

http://youtu.be/BNXkdmwjM_4?t=44s V's http://youtu.be/sV1vRdO3A6c?t=45s

Specific Examples of Complex Numbers

Engineers use complex numbers in analysing stresses and strains on beams and in studying resonance phenomena in structures as different as tall buildings and suspension bridges. The complex numbers come up when they look for the eigenvalues and eigenvectors of a matrix.

Complex numbers are used in studying the flow of fluids around obstacles, such as the flow around a pipe.

Adding and Subtracting Complex Numbers

Group the real parts together and group the imaginary parts together.

Examples:

(a) $(3-i)+(2+3i) = 3+2-i+3i$

$= 5+2i$

(b) $(5+3i)-(3-2i) = 5-3+3i+2i$

$= 2+5i$

Multiplication of Complex Numbers

General approach:

- Multiply as normal
- Convert i^2 to -1, i^3 to $-i$, etc.
- Group real parts together and group imaginary parts together.

Examples of Multiplication

(a) $(3+2i)(3-2i) = 9-6i+6i-4i^2$

$= 9-4i^2$

$= 9-4(-1) = 13$

(b) $(3+2i)^2 = 9+6i+6i+4i^2$

$= 9+12i+4(-1)$

$= 5+12i$

The Complex Conjugate

Notice in Example (a) that the product of two complex numbers can be a real number.

This occurs with pairs of complex numbers of the form $a+bi$ and $a-bi$ called complex conjugates.

If $\quad z = a+bi$

Then $\quad \bar{z} = a-bi$

Example of Division

Simplify $\frac{2+3i}{4-2i}$ into the form $a+bi$

$$\frac{2+3i}{4-2i} = \frac{2+3i}{4-2i}\left(\frac{4+2i}{4+2i}\right) = \frac{8+4i+12i+6i^2}{16-4i^2}$$

$$= \frac{8-6+16i}{16+4} = \frac{1}{20}(2+16i)$$

$$= \frac{1}{10}+\frac{4}{5}i$$

Dividing one Complex Number by Another

General Approach:

Multiply above and below by the complex conjugate of the denominator.

This will ensure that the denominator becomes a constant, thereby making it much easier to obtain an answer in the form $a + bi$

Examples of the use of Complex Numbers

Solve $3x^2 - 2x + 5 = 0$.

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(5)}}{2(3)}$$

$$= \frac{2 \pm \sqrt{-56}}{6} \qquad = \frac{2 \pm 2\sqrt{14}i}{6}$$

$$= \frac{1}{3} \pm \frac{\sqrt{14}}{3}i$$

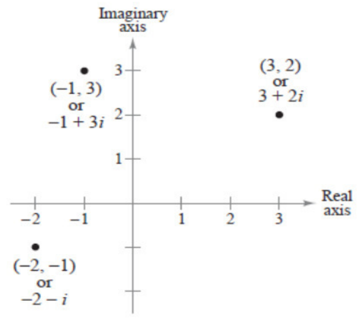
Argand Diagram

Just as real numbers can be represented by points on the real number line, you can represent a complex number $z = a + bi$ at the point (a, b) in a coordinate plane (the complex plane).

This is known as an *Argand Diagram*.

The horizontal axis is called the real axis and the vertical axis is called the imaginary axis.

Argand Diagram



Modulus of a Complex Number

The modulus (or absolute value) of the complex number $z = a + bi$ is defined as the distance between the origin (0,0) and the point (a, b)

$$z = a + bi$$

Modulus: $|z| = \sqrt{a^2 + b^2}$

The modulus can be represented by:

$|z|$ **or** r **or** $|a + bi|$

Modulus of a Complex Number

Example: Find the modulus of the complex number $3 - 4i$

Solution:

$$r = \sqrt{(3)^2 + (-4)^2}$$

$$r = \sqrt{25} = 5$$

The modulus of the complex number $3 - 4i$ is 5

This means that the distance from the origin to $(3, -4)$ is 5 units.

Modulus of a Complex Number

Example: Find the modulus of $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$

Solution:

$$r = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2}$$

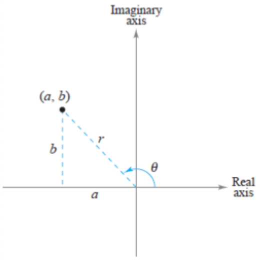
$$r = \sqrt{\frac{1}{2} + \frac{1}{2}} \quad = \ 1$$

The modulus of the complex number $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$ is 1

Polar Form of a Complex Number

The Modulus (or absolute value) is needed to write a complex number in **Polar** form

$$a + bi = r(Cos\theta + iSin\theta)$$



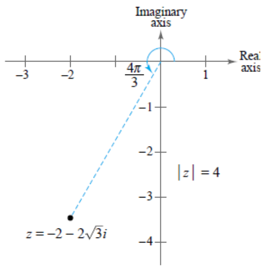
Polar Form of a Complex Number

θ is the angle between the line representing the complex number and the positive side of the real axis.

To write a complex number in polar form, we need to find the values of r and θ, then sub them into the formula.

Converting a Complex Number to Polar Form

Write the complex number $z = -2 - 2\sqrt{3}i$ in polar form.



Converting a Complex Number to Polar Form

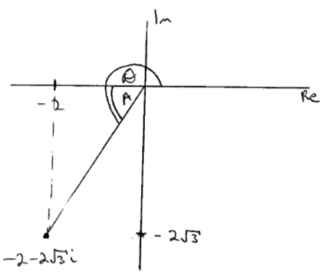
Write the complex number $z = -2 - 2\sqrt{3}i$ in polar form.

Solution: First, find the modulus, r, of the complex number:

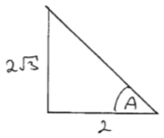
$$r = |-2 - 2\sqrt{3}i| = \sqrt{(-2)^2 + (-2\sqrt{3})^2}$$

$$r = \sqrt{4 + 12} = 4$$

Finding θ



Finding θ

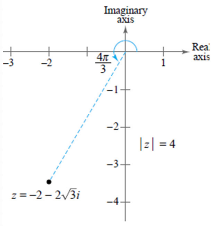


$$\tan A = \frac{2\sqrt{3}}{2} = \sqrt{3}$$

$$A = \tan^{-1}\sqrt{3} = 60°$$

$$\Rightarrow \theta = 180° + 60° = 240°$$

Converting a Complex Number to Polar Form



Angle $\theta = 240°$

$r = 4$

$a + bi = r(Cos\theta + iSin\theta)$

$-2 - 2\sqrt{3}i = 4(Cos240° + iSin240°)$

The polar form of $-2 - 2\sqrt{3}i$ is $4(Cos240° + iSin240°)$

Steps for Finding the Polar Form of a Complex Number

1. Find the modulus of the complex number.
2. Draw the complex number on an Argand Diagram and find the angle the complex number makes with the positive side of the real axis.
3. Use this information to rewrite the complex number in the form $r(Cos\theta + iSin\theta)$

De Moivre's Theorem

$$z = r(\cos\theta + i\sin\theta)$$

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

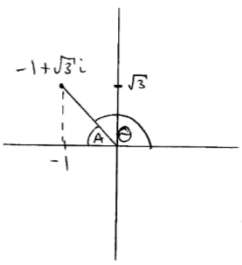
Proof: http://math.ucsd.edu/~wgarner/math4c/derivations/trigidentities/demoivrethm.htm

Example: Use De Moivre's Theorem to find

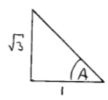
$$(-1 + \sqrt{3}i)^{12}$$

Step 1: Convert $-1 + \sqrt{3}i$ to polar form.

De Moivre's Theorem – Ex. 1



De Moivre's Theorem – Ex. 1



$$\tan A = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$A = \tan^{-1}\sqrt{3} = 60°$$

$$\theta = 180° - 60° = 120° \text{ or } \frac{2\pi}{3}$$

De Moivre's Theorem – Ex. 1

Angle: $\theta = 120°$

$$r = \sqrt{(-1)^2 + (\sqrt{3})^2} = 2$$

$$a + bi = r(Cos\theta + iSin\theta)$$

$$-1 + \sqrt{3}i = 2(Cos120° + iSin120°)$$

De Moivre's Theorem – Ex. 1

$$-1 + \sqrt{3}i = 2(Cos120° + iSin120°)$$

$$\left(-1 + \sqrt{3}i\right)^{12} = 2^{12}(Cos12(120°) + iSin12(120°))$$

$$\left(-1 + \sqrt{3}i\right)^{12} = 4096(Cos1440° + iSin1440°)$$

$$\left(-1 + \sqrt{3}i\right)^{12} = 4096(1 + i(0))$$

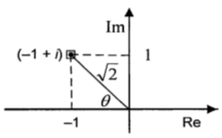
$$\left(-1 + \sqrt{3}i\right)^{12} = 4096$$

De Moivre's Theorem - Example 2

Q. Let $z = -1 + i$ Use De Moivre's theorem to evaluate z^5 and z^9

Solution: Write z in polar form first.

$$z = -1 + i$$



$\tan\theta = \dfrac{1}{1} = 1$ $\qquad \theta = 45°$

Actual angle: $180° - 45° = 135°$

De Moivre's Theorem - Example 2

$$z = -1 + i$$

Modulus: $r = \sqrt{a^2 + b^2} = \sqrt{(-1)^2 + (1)^2} = \sqrt{2}$

We now know: Angle $= 135°$ Modulus $= \sqrt{2}$

$$a + bi = r(Cos\theta + iSin\theta)$$

$$-1 + i = \sqrt{2}(Cos135° + iSin135°)$$

$$-1 + i = 2^{0.5}\,(Cos135° + iSin135°)$$

De Moivre's Theorem - Example 2

Once the complex number z is in polar form, we can use De Moivre's theorem to find z^5

$$z = 2^{0.5}\,(Cos135° + iSin135°)$$

$$\boxed{\begin{array}{c} z = r(\cos\theta + i\sin\theta) \\ z^n = r^n(\cos n\theta + i\sin n\theta) \end{array}}$$

$$z^5 = (2^{0.5})^5\,(Cos(5)(135°) + iSin(5)(135°))$$

De Moivre's Theorem - Example 2

$$z^5 = (2^{0.5})^5\,(Cos(5)(135°) + iSin(5)(135°))$$

$$z^5 = 2^{2.5}(Cos675° + iSin675°)$$

$$z^5 = \sqrt{32}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = 4 - 4i$$

$$z^5 = 4 - 4i$$

De Moivre’s Theorem - Example 2

We can also use De Moivre’s theorem to find z^9

$$z = r(\cos\theta + i\sin\theta)$$

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

$$z = -1 + i = 2^{0.5}\ (Cos135° + iSin135°)$$

$$z^9 = (2^{0.5})^9\ (Cos(9)(135°) + iSin(9)(135°))$$

De Moivre’s Theorem - Example 2

$$z = 2^{0.5}\ (Cos135° + iSin135°)$$

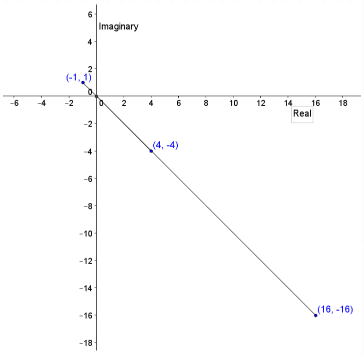
$$z^9 = (2^{0.5})^9\ (Cos(9)(135°) + iSin(9)(135°))$$

$$z^9 = 2^{4.5}(Cos1215° + iSin1215°)$$

$$z^9 = \sqrt{512}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) = -16 + 16i$$

$$z^9 = -16 + 16i$$

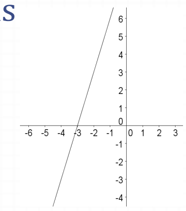
De Moivre’s Theorem - Example 2



Roots of Equations

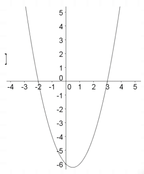
1. A linear equation has one root

e.g. $y = 3x + 2$



2. A quadratic equation typically has two

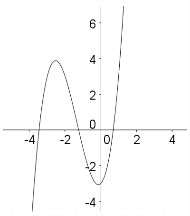
e.g. $y = 4x^2 - 3x + 7$



Roots of Equations

3. A cubic equation typically has three roots

e.g. $y = x^3 + 4x^2 + x - 3$



In general, an equation of degree n typically has n roots.

Roots of a Complex Number

Q. Find the four complex numbers z such that

$$z^4 = -8 - 8\sqrt{3}i$$

Note: This is an equation of degree 4, that’s why it has 4 roots

Solution: Write z^4 in polar form (find modulus and angle etc.).

$$-8 - 8\sqrt{3}i = 16(Cos240° + iSin240°)$$

$$z^4 = 16(Cos240° + iSin240°)$$

Roots of a Complex Number

$$z^4 = 16(Cos240° + iSin240°)$$

$$z = [16(Cos240° + iSin240°)]^{\frac{1}{4}}$$

$$z = 16^{\frac{1}{4}}\left(Cos\left(\frac{1}{4}\right)(240° + n360°) + iSin\left(\frac{1}{4}\right)(240° + n360°)\right)$$

Here, we have applied De Moivre's theorem and we will add $n360°$ to the angle with $n = 0, 1, 2, 3$.

Adding $n360°$ to the angle does not change the angle as it's just adding a number of full rotations but it does ensure that we will obtain 4 answers (4 roots) rather than just one.

Roots of a Complex Number

$$z = 16^{\frac{1}{4}}\left(Cos\left(\frac{1}{4}\right)(240° + n360°) + iSin\left(\frac{1}{4}\right)(240° + n360°)\right)$$

$n = 0$

$$z = 16^{\frac{1}{4}}\left(Cos\left(\frac{1}{4}\right)(240°) + iSin\left(\frac{1}{4}\right)(240°)\right)$$

$$z = 2(Cos\ 60° + iSin60°) = 1 + \sqrt{3}i$$

Roots of a Complex Number

$$z = 16^{\frac{1}{4}}\left(Cos\left(\frac{1}{4}\right)(240° + n360°) + iSin\left(\frac{1}{4}\right)(240° + n360°)\right)$$

$n = 1$

$$z = 16^{\frac{1}{4}}\left(Cos\left(\frac{1}{4}\right)(600°) + iSin\left(\frac{1}{4}\right)(600°)\right)$$

$$z = 2(Cos\ 150° + iSin\ 150°) = -\sqrt{3} + i$$

Roots of a Complex Number

$$z = 16^{\frac{1}{4}}\left(Cos\left(\frac{1}{4}\right)(240° + n360°) + iSin\left(\frac{1}{4}\right)(240° + n360°)\right)$$

$n = 2$

$$z = 16^{\frac{1}{4}}\left(Cos\left(\frac{1}{4}\right)(960°) + iSin\left(\frac{1}{4}\right)(960°)\right)$$

$$z = 2(Cos\ 240° + iSin\ 240°) = -1 - \sqrt{3}i$$

Roots of a Complex Number

$$z = 16^{\frac{1}{4}}\left(Cos\left(\frac{1}{4}\right)(240° + n360°) + iSin\left(\frac{1}{4}\right)(240° + n360°)\right)$$

$n = 3$

$$z = 16^{\frac{1}{4}}\left(Cos\left(\frac{1}{4}\right)(1320°) + iSin\left(\frac{1}{4}\right)(1320°)\right)$$

$$z = 2(Cos\ 330° + iSin\ 330°) = -\sqrt{3} - i$$

Roots of a Complex Number

Roots: $1 + \sqrt{3}i\ ; -\sqrt{3} + i\ ;\ -1 - \sqrt{3}i\ ;\ -\sqrt{3} - i$

The four complex numbers z such that

$$z^4 = -8 - 8\sqrt{3}i$$

are:

$$z = 1 + \sqrt{3}i\ ; -\sqrt{3} + i\ ;\ -1 - \sqrt{3}i\ ;\ -\sqrt{3} - i$$

Roots of a Complex Number

Q. Find the three complex numbers z such that

$$z^3 = -1 + i$$

Note: This is an equation of degree 3 so it has 3 roots.

Solution: Write z^3 in polar form (find modulus and angle etc.).

$$-1 + i = \sqrt{2}\,(Cos135° + iSin135°)$$

$$z^3 = 2^{\frac{1}{2}}\,(Cos135° + iSin135°)$$

Roots of a Complex Number

$$z^3 = 2^{\frac{1}{2}}\,(Cos135° + iSin135°)$$

$$z = \left[2^{\frac{1}{2}}\,(Cos135° + iSin135°)\right]^{\frac{1}{3}}$$

$$z = \left(2^{\frac{1}{2}}\right)^{\frac{1}{3}}\left(Cos\left(\frac{1}{3}\right)(135° + n360°) + iSin\left(\frac{1}{3}\right)(135° + n360°)\right)$$

Here, we have applied De Moivre's theorem and we will add $n360°$ to the angle with $n = 0, 1, 2$.

Adding $n360°$ to the angle does not change the angle as it's just adding a number of full rotations but it does ensure that we will obtain 3 answers (3 roots) rather than just one.

Roots of a Complex Number

$$z = 2^{\frac{1}{6}}\left(Cos\left(\frac{1}{3}\right)(135° + n360°) + iSin\left(\frac{1}{3}\right)(135° + n360°)\right)$$

$n = 0$

$$z = 2^{\frac{1}{6}}\left(Cos\left(\frac{1}{3}\right)(135°) + iSin\left(\frac{1}{3}\right)(135°)\right)$$

$$z = 1.122(Cos\ 45° + iSin45°) = 0.8 + 0.8i$$

Roots of a Complex Number

$$z = 2^{\frac{1}{6}}\left(Cos\left(\frac{1}{3}\right)(135° + n360°) + iSin\left(\frac{1}{3}\right)(135° + n360°)\right)$$

$n = 1$

$$z = 2^{\frac{1}{6}}\left(Cos\left(\frac{1}{3}\right)(495°) + iSin\left(\frac{1}{3}\right)(495°)\right)$$

$$z = 1.122(Cos\ 165° + iSin165°) = -1.08 + 0.3i$$

Roots of a Complex Number

$$z = 2^{\frac{1}{6}}\left(Cos\left(\frac{1}{3}\right)(135° + n360°) + iSin\left(\frac{1}{3}\right)(135° + n360°)\right)$$

$n = 2$

$$z = 2^{\frac{1}{6}}\left(Cos\left(\frac{1}{3}\right)(855°) + iSin\left(\frac{1}{3}\right)(855°)\right)$$

$$z = 1.122(Cos\ 285° + iSin285°) = 0.3 - 1.08i$$

Roots of a Complex Number

Roots: $0.8 + 0.8i$; $-1.08 + 0.3i$; $0.3 - 1.08i$

The three complex numbers z such that

$$z^3 = -1 + i$$

are:

$$z = 0.8 + 0.8i\,;\ -1.08 + 0.3i\,;\ 0.3 - 1.08i$$

The Conjugate Root Theorem

If the complex number $z = a + bi$, where $a, b \epsilon R$, is a root of the polynomial $f(z)$ with real coefficients, then $\bar{z} = a - bi$ (the conjugate of z) is also a root.

Example: If $z = 4 + 3i$ is a root of the polynomial $f(z) = z^2 - 8z + 25$ then $\bar{z} = 4 - 3i$ is also a root.

Example: If $z = 1 - 2i$ is a root of the polynomial $f(z) = 2z^3 - 7z^2 + 16z - 15$ then $\bar{z} = 1 + 2i$ is also a root.

Explanation:
http://en.wikipedia.org/wiki/Complex_conjugate_root_theorem

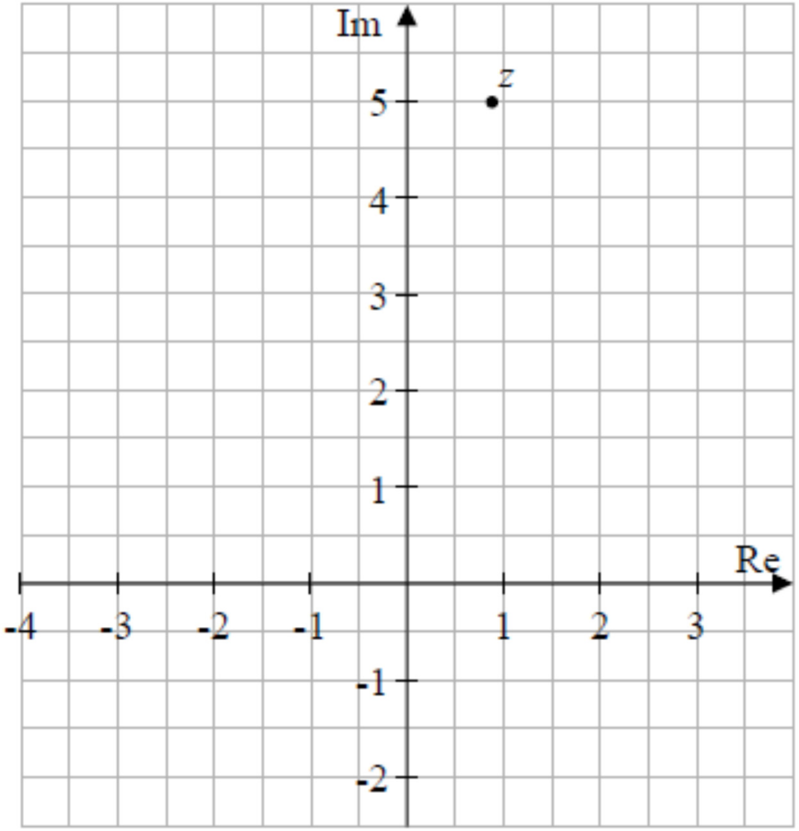
2012

Question 3

The complex number z has modulus $5\frac{1}{16}$ and argument $\dfrac{4\pi}{9}$.

(a) Find, in polar form, the four complex fourth roots of z.
(That is, find the four values of w for which $w^4 = z$.)

(b) z is marked on the Argand diagram below.
On the same diagram, show the four answers to part **(a)**.



2011

Question 2

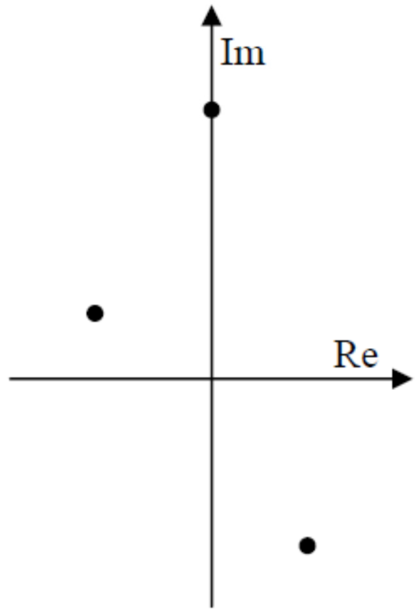
(a) **(i)** Write the complex number $1-i$ in polar form.

(ii) Use De Moivre's theorem to evaluate $(1-i)^9$, giving your answer in rectangular form.

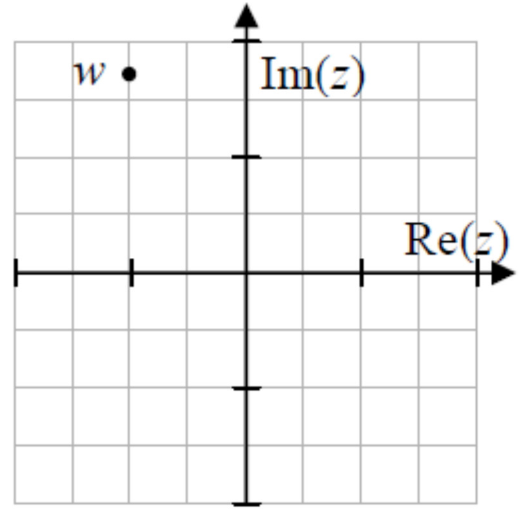
(b) A complex number z has modulus greater than 1.
The three numbers z, z^2, and z^3 are shown on the Argand diagram.
One of them lies on the imaginary axis, as shown.

(i) Label the points on the diagram to show which point corresponds to which number.

(ii) Find θ, the argument of z.



Question 1



(a) $w = -1 + \sqrt{3}i$ is a complex number, where $i^2 = -1$.

(i) Write w in polar form.

(ii) Use De Moivre's theorem to solve the equation $z^2 = -1 + \sqrt{3}i$, giving your answer(s) in rectangular form.

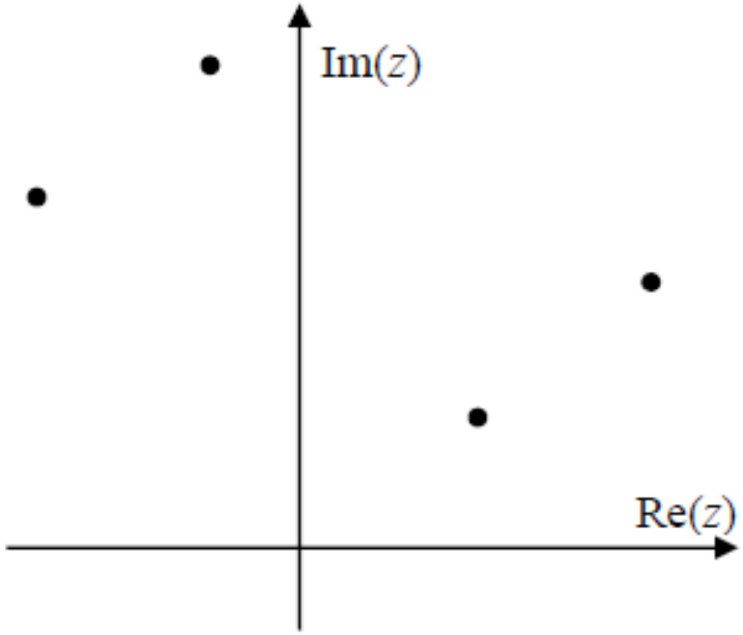
(b) Four complex numbers z_1, z_2, z_3 and z_4 are shown on the Argand diagram. They satisfy the following conditions:

$z_2 = iz_1$

$z_3 = kz_1$, where $k \in \mathbb{R}$

$z_4 = z_2 + z_3$.

The same scale is used on both axes.



(i) Identify which number is which, by labelling the points on the diagram.

(ii) Write down the approximate value of *k*.

Answer: ______________

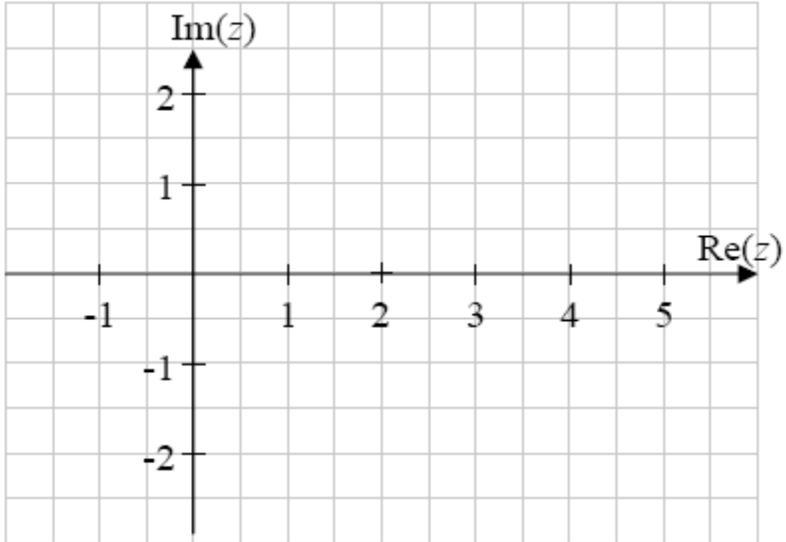
2014

Question 2

Let $z_1 = 1 - 2i$, where $i^2 = -1$.

(a) The complex number z_1 is a root of the equation $2z^3 - 7z^2 + 16z - 15 = 0$.

Find the other two roots of the equation.



(b) **(i)** Let $w = z_1\overline{z}_1$, where $\overline{z}_1$ is the conjugate of z_1. Plot z_1, $\overline{z}_1$ and w on the Argand diagram and label each point.

(ii) Find the measure of the acute angle, $\overline{z}_1 w z_1$, formed by joining $\overline{z}_1$ to w to z_1 on the diagram above. Give your answer correct to the nearest degree.