Algebra 1 – Equations, Infinity, Inequalities, and Indices

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Solving Mathematical Equations

What you do to one side of a mathematical equation, you must do the exact same to the other side.

Changing sign

Why does the sign of a value change when you move it to the other side of the equals sign? For example:

$$x^2 + 5 = 30$$

$$x^2 = 30 - 5$$

Changing Sign

Explanation:

$$x^2 + 5 = 30$$

We actually subtract 5 from both sides (what you do to one side, you do the same to the other):

$$x^2 + 5 - 5 = 30 - 5$$

$$x^2 = 30 - 5$$

It's just quicker to use the shortcut of changing the sign when you move the value to the other side of the equals sign.

Solving Mathematical Equations

Example:

Solve for x in the following

$$\frac{(2x-1)^3}{5} = 25$$

Solving Mathematical Equations

Example:

Solve for x in the following

$$\frac{3x+7}{11} + 3 = 5$$

Rules of Indices

1. $a^m \times a^n = a^{m+n}$

e.g. $5^3 \times 5^4 = 5^7$

2. $\frac{a^m}{a^n} = a^{m-n}$

e.g. $\frac{2^9}{2^4} = 2^5$

Why?

Rules of Indices

3. $(a^m)^n = a^{mn}$

e.g. $(10^2)^3 = 10^6$

4. $a^0 = 1$

e.g. $10^0 = 1$ $783^0 = 1$ $x^0 = 1$

Why?

Rules of Indices

5. $\frac{1}{a^m} = a^{-m}$

e.g. $\frac{1}{8^2} = 8^{-2}$ $\frac{1}{x^{-3}} = x^3$

6. $\sqrt[n]{a} = a^{\frac{1}{n}}$

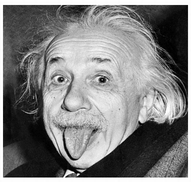
e.g. $\sqrt[3]{64} = 64^{\frac{1}{3}}$ $\sqrt[5]{32} = 32^{\frac{1}{5}}$

Infinity

"Two things are infinite: the universe and human stupidity; and I'm not sure about the universe."

– Albert Einstein

∞



Infinity

"It has been theorized that an infinite number of monkeys banging on an infinite number of typewriters would eventually reproduce the written works of Shakespeare. Thanks to the Internet, we now know this is not true."

~ Kurt Vonnegut

Actual Experiment: http://www.bbc.co.uk/news/technology-15060310
Karl Pilkington: http://www.youtube.com/watch?v=kvZ0PdZ3p7E
Simpsons: https://www.youtube.com/watch?v=no_elVGGgW8

Infinity

If you divide a number by infinity, what will this equal?

$$\frac{any\ number}{\infty} = ?$$

$$\frac{any\ number}{\infty} = 0$$

e.g. $\frac{12}{\infty} = 0 \quad \frac{1{,}027{,}593}{\infty} = 0 \quad -\frac{58{,}772{,}930}{\infty} = 0$

Infinity

Similarly, if you divide any number by zero, your answer will be infinity:

$$\frac{any\ number}{0} = \infty$$

e.g. $\frac{12}{0} = \infty \quad \frac{1{,}027{,}593}{0} = \infty \quad \frac{58{,}772{,}930}{0} = \infty$

Inequalities

- Legally allowed to drive in Ireland:
  Age $\geq$ 17 years
- Beat the World Record for 100m sprint:
  Time $< 9.58\ seconds$
- If x represents the age (in years) you must be to be eligible to join the Royal Navy then:

$$16 \leq x \leq 37$$

Inequalities

Sign	Meaning	e.g.
$<$	Less than	$2 < 4$
$>$	Greater than	$5 > 3$
$\leq$	Less than or equal to	$\beta \leq 7$
$\geq$	Greater than or equal to	$a \geq b$

Rules for Solving Inequalities

1. Multiplication or division of both sides of an inequality by a negative number will change the sign of the inequality (e.g. from $>$ to $<$)
2. Inverting fractions will change the inequality sign.

Otherwise, same rules for solving equations apply to solving inequalities.

Solving Inequalities

Solve for x in the following: $10 - 4x \geq 2$

Solution: $10 - 4x \geq 2$

$-4x \geq -8$ [Subtract 10 from both sides]

$x \leq 2$ [Divide both sides by –4]

Solving Inequalities

Solve for a in the following: $-8 - \frac{3a}{2} < 1$

Solution: $-8 - \frac{3a}{2} < 1$

$-\frac{3a}{2} < 9$ [Add 8 to both sides]

$-3a < 18$ [Multiply both sides by 2]

$a > -6$ [Divide both sides by –3]

Solving Inequalities

Solve for t in the following: $10 \geq 6 + 8t$

Solution: $10 \geq 6 + 8t$

$4 \geq 8t$ [Subtract 6 from both sides]

$\frac{1}{2} \geq t$ [Divide both sides by 8]

$t \leq \frac{1}{2}$

Quadratic Equations

Examples:

$y = 3x^2 + 2x - 7$ $\quad$ $y = x^2 - 7x + 3$

$y = -2x^2 + x + 3$ $\quad$ $y = -5x^2 + 4x + 9$

Quadratic equations are of the form:

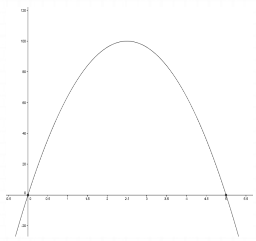
$y = ax^2 + bx + c$

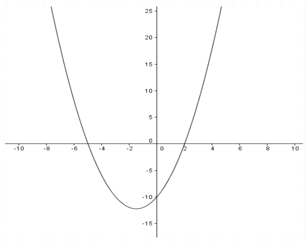
Quadratic Equations

- Quadratic equations are used in many areas of science and engineering.
- The path of a projectile (e.g. a cannon ball) is parabolic, and we can use a quadratic equation to find out where the projectile is going to land.

http://www.youtube.com/watch?v=aSwcw6v7TMA

Quadratic Equations – Shape

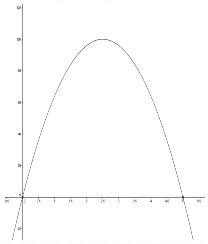




Quadratic Equations – Shape

- When the graph of a quadratic equation is drawn, it will take on either an "n-shape" or a "u-shape".

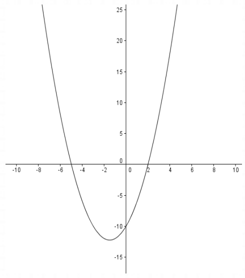
- It will be "n-shaped" if the x^2 coefficient of the equation is negative.



e.g. $y = -3x^2 + 7x - 10$

Quadratic Equations – Shape

- The curve will be "u-shaped" if the x^2 coefficient of the equation is positive



e.g. $y = 5x^2 - 2x - 11$

U-Shaped or n-shaped Curve?

1. $y = 3x^2 - 5x + 3$ *u*
2. $y = -x^2 + x + 7$ *n*
3. $y = -12a^2 + 13a - 21$ *n*
4. $y = 4\beta^2 - 3\beta + 6$ *u*

Solving Quadratic Inequalities

Solve $x^2 + 3x - 10 \geq 0$

Solution: Initially, ignore the inequality and solve.

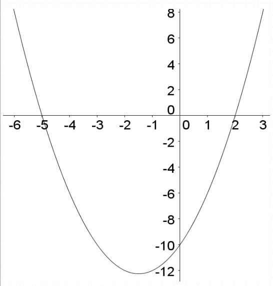
$$x^2 + 3x - 10 = 0$$

$$(x+5)(x-2) = 0$$

Roots of the quadratic: $x = -5$ $x = 2$

Now sketch the graph of $y = x^2 + 3x - 10$

Graph of $y = x^2 + 3x - 10$



When is $x^2 + 3x - 10 \geq 0$?

We can clearly see that $x^2 + 3x - 10 \geq 0$ when:

$x \leq -5$ and when $x \geq 2$

Answer: $x \leq -5$ or $x \geq 2$

Solving Quadratic Inequalities

Solve $4x^2 + 11x - 3 < 0$

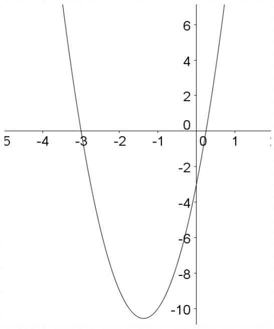
Solution: Ignore the inequality and solve

$$(4x-1)(x+3) = 0$$

Roots of the quadratic: $x = \frac{1}{4}$ $x = -3$

Now sketch the graph of $y = 4x^2 + 11x - 3$

Graph of $y = 4x^2 + 11x - 3$



When is $4x^2 + 11x - 3 < 0$?

We can clearly see that $4x^2 + 11x - 3 < 0$ when:

$x > -3$ and when $x < \frac{1}{4}$

Answer: $x > -3$ or $x < \frac{1}{4}$

Solving Quadratic Inequalities

Solve $x^2 - 16 > 0$

Solution: Ignore the inequality and solve

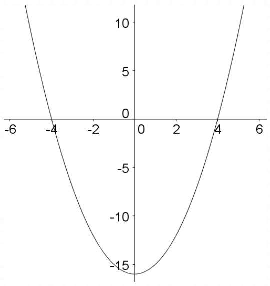
$$(x - 4)(x + 4) = 0$$

Roots of the quadratic: $x = 4$ $x = -4$

Now sketch the graph of $y = x^2 - 16$

Graph of $y = x^2 - 16$

When is $x^2 - 16 > 0$?



We can clearly see that $x^2 - 16 > 0$ when:

$x < -4$ and when $x > 4$

Answer: $x < -4$ or $x > 4$

Steps for Solving Quadratic Inequalities

1. Ignore the inequality at the start (replace with equals sign) and find values for x (these values will be the roots of the quadratic).
2. Sketch the graph of the quadratic using the roots.
3. Determine what values for x satisfy the original inequality.

Solving Quadratic Inequalities – Further Example

Solve the following inequality:

$$\frac{x}{2x - 1} < -2$$

Method: Turn the rational inequality into a quadratic inequality by multiplying both sides by a positive expression.

$$(2x - 1)^2 \frac{x}{2x - 1} < -2(2x - 1)^2$$

Note: multiplying both sides by a squared value ensures that the inequality sign is not affected.

Solving Quadratic Inequalities – Further Example

Complete all multiplication and tidy up the expression

$$(2x - 1)(x) < -2(2x - 1)(2x - 1)$$

$$2x^2 - x < -2(4x^2 - 4x + 1)$$

$$2x^2 - x < -8x^2 + 8x - 2$$

$$10x^2 - 9x + 2 < 0$$

Solving Quadratic Inequalities – Further Example

Solve the Quadratic to find the roots so that we can sketch the graph of the quadratic.

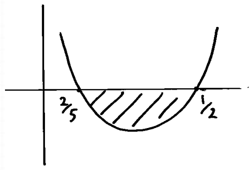
$$(5x - 2)(2x - 1) = 0$$

$5x - 2 = 0$	$2x - 1 = 0$
$x = \frac{2}{5}$	$x = \frac{1}{2}$

Solving Quadratic Inequalities – Further Example

Roots: $x = \frac{2}{5}$ $x = \frac{1}{2}$

When is $10x^2 - 9x + 2 < 0$?



Answer: $\frac{2}{5} < x < \frac{1}{2}$

Simultaneous Equations

Solve the following set of simultaneous equations:

$$2x + 3y = 0$$
$$x + y + z = 0$$
$$3x + 2y - 4z = 9$$

Step 1: Eliminate one of the variables.

$$x + y + z = 0$$
$$3x + 2y - 4z = 9$$

$$4x + 4y + 4z = 0$$
$$\underline{3x + 2y - 4z = 9}$$
$$7x + 6y = 9$$

Simultaneous Equations

Step 2: Solve for either x or y using the following equations:

$$7x + 6y = 9$$
$$2x + 3y = 0$$

$$7x + 6y = 9$$
$$\underline{-4x - 6y = 0}$$
$$3x = 9$$

$$x = 3$$

Step 3: Solve for y by subbing for x in the equation:

$$2x + 3y = 0$$
$$2(3) + 3y = 0$$
$$6 + 3y = 0$$
$$y = -2$$

Simultaneous Equations

Step 4: Solve for z using one of the original equations.

$$x + y + z = 0$$

We know:

$x = 3$ and $y = -2$

$$3 + (-2) + z = 0$$
$$z = -1$$

Answers:

$x = 3$
$y = -2$
$z = -1$

Simultaneous Equations

Method:

- Select one pair of equations and eliminate one of the variables.
- Select another pair and eliminate the same variable.
- Solve these two new equations simultaneously.
- Use answers to find third variable.

Simultaneous Equations – ex. 2

Solve the following set of simultaneous equations:

$$a + 2b + 1 = 0$$
$$a^2 - ab + b^2 = 3$$

We need to take a different approach here.

Find a value for a or b from the first equation and sub that into the second equation, then solve.

From equation 1 we know:

$$a = -2b - 1$$

Sub for a in equation 2:

$$(-2b-1)^2 - (-2b-1)b + b^2 = 3$$

$$4b^2 + 4b + 1 + 2b^2 + b + b^2 = 3$$

$$7b^2 + 5b - 2 = 0$$

Simultaneous Equations – ex. 2

We can now solve this new equation to find values for b

$$7b^2 + 5b - 2 = 0$$

$$(7b - 2)(b + 1) = 0$$

$7b - 2 = 0$	$b + 1 = 0$
$b = \frac{2}{7}$	$b = -1$

Now we need to find a

We know:
$$a = -2b - 1$$

When $b = -1$

$$a = -2(-1) - 1$$

$$a = 1$$

Simultaneous Equations – ex. 2

When $b = \frac{2}{7}$

$$a = -2\left(\frac{2}{7}\right) - 1$$

$$a = -\frac{11}{7}$$

Answers:

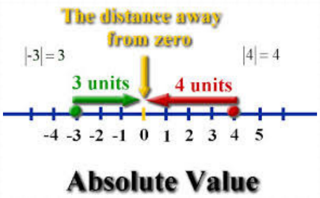
$a = -\frac{11}{7}$ $\quad b = \frac{2}{7}$

$a = 1$ $\quad b = -1$

Modulus

The Absolute Value or Modulus of a real number is the magnitude of the number without regard to its sign.

It could also be considered as the distance the number is from zero:



Modulus – ex. 1

Solve for x in the following:

$$|2x - 1| \leq 3$$

Solution: Square both sides

$$(2x-1)^2 \leq 9$$

Complete all multiplication and tidy up the expression:

$$(2x-1)^2 \leq 9$$

$$4x^2 - 4x + 1 \leq 9$$

$$4x^2 - 4x - 8 \leq 0$$

$$x^2 - x - 2 \leq 0$$

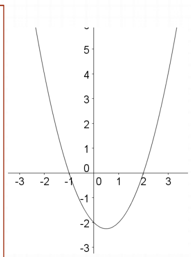
Modulus – ex. 1

Solve the quadratic to find the roots and sketch the curve:

$$x^2 - x - 2 = 0$$

$$(x-2)(x+1) = 0$$

$$x = 2 \qquad x = -1$$



Where is $x^2 - x - 2 \leq 0$?

Answer: $-1 \leq x \leq 2$

The inequality $|2x - 1| \leq 3$ is true when

$$-1 \leq x \leq 2$$

Modulus – ex. 1 Alternative Approach

Solve for x in the following:

$$|2x - 1| \leq 3$$

Solution: We know that the value for $2x - 1$ must be, at most, a distance 3 units from zero on the number line.

Therefore:

$$2x - 1 \leq 3$$

Or

$$2x - 1 \geq -3$$

$$2x - 1 \leq 3$$

$$2x \leq 4$$

$$x \leq 2$$

$$2x - 1 \geq -3$$

$$x \geq -1$$

Answer: $-1 \leq x \leq 2$

Factor Theorem

Polynomials are of the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

where all powers are non-negative whole numbers and the a's are constants.

Examples:

$f(x) = 5x^2 - 2x + 8$ $\qquad$ $g(x) = -x^4 + 2x^3 + 5x + 1$

Example of non-polynomial:

$$h(x) = 5x^3 + 3\sqrt{x} + 7$$

Factor Theorem – ex. 1

A Polynomial $f(x)$ has a factor $(x - a)$ iff (if and only if) $f(a) = 0$

Example: $\qquad f(x) = 2x^3 - 3x^2 - 17x + 30$

We can use factors of the constant value (30) in $f(x)$ to help estimate the initial root. After some trial and error, we find $x = -3$ is a root.

$$f(-3) = 2(-3)^3 - 3(-3)^2 - 17(-3) + 30$$

$$f(-3) = 0$$

Thus $x = -3$ is a root of $f(x)$ and $(x + 3)$ is a factor of $f(x)$

Factor Theorem – ex. 1

To find the other two factors, we divide $f(x)$ by the factor we know $(x + 3)$

$$\frac{2x^3 - 3x^2 - 17x + 30}{x + 3} = 2x^2 - 9x + 10$$

Factorise $2x^2 - 9x + 10$ to find the remaining factors of $f(x)$

$$(2x - 5)(x - 2)$$

Now we know that $f(x)$ can be factorised as follows:

$$2x^3 - 3x^2 - 17x + 30 = (x + 3)(2x - 5)(x - 2)$$

Factor Theorem – Why does it work?

Using factors of the constant value in our cubic polynomials to find one of the roots works because, if we multiply out the factors we found, the constants in each factor will multiply to produce the resulting constant in the polynomial.

We now know:

$$2x^3 - 3x^2 - 17x + 30 = (x + 3)(2x - 5)(x - 2)$$

$$2x^3 - 3x^2 - 17x + 30 = (x + 3)(2x^2 - 9x + 10)$$

$$2x^3 - 3x^2 - 17x + 30 = 2x^3 - 3x^2 - 17x + 30$$

Factor Theorem – ex. 2

A cubic function f is defined for $x \in \mathbb{R}$ as

$$f : x \mapsto x^3 + (1 - k^2)x + k, \text{ where } k \text{ is a constant.}$$

(a) Show that $-k$ is a root of f.

(b) Find, in terms of k, the other two roots of f.

(c) Find the set of values of k for which f has exactly one real root.

Factor Theorem – ex. 2

(a) If $-k$ is a root then $f(-k) = 0$

$$f(x) = x^3 + (1 - k^2)x + k$$

$$f(-k) = (-k)^3 + (1 - k^2)(-k) + k$$

$$f(-k) = -k^3 - k + k^3 + k$$

$$f(-k) = 0$$

Conclusion: $-k$ is a root of $f(x) = x^3 + (1 - k^2)x + k$

Factor Theorem – ex. 2

(b) Find, in terms of k, the other two roots of f.

If $x = -k$ is a root then $x + k$ is a factor of

$$x^3 + (1 - k^2)x + k$$

Solution: Divide $x + k$ into $x^3 + (1 - k^2)x + k$

$$\frac{x^3 + (1 - k^2)x + k}{x + k} = x^2 - kx + 1$$

Factor Theorem – ex. 2

We now know:

$$x^3 + (1 - k^2)x + k = (x + k)(x^2 - kx + 1)$$

Solve $x^2 - kx + 1$ to find final two roots

Use $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$$a = 1 \quad b = -k \quad c = 1$$

Factor Theorem – ex. 2

$$x = \frac{-(-k) \pm \sqrt{(-k)^2 - 4(1)(1)}}{2(1)}$$

$$x = \frac{k \pm \sqrt{k^2 - 4}}{2}$$

Roots: $x = -k \qquad x = \frac{k + \sqrt{k^2 - 4}}{2} \qquad x = \frac{k - \sqrt{k^2 - 4}}{2}$

Factor Theorem – ex. 2

(c) Find the set of values of k for which f has exactly one real root.

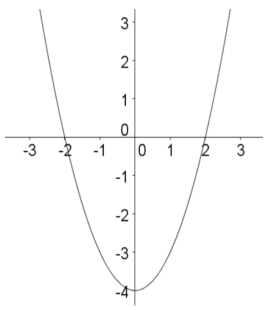
The real root must be $x = -k$ as we are told at the start that $k \in R$

Thus $\quad x = \frac{k + \sqrt{k^2 - 4}}{2} \quad and \quad x = \frac{k - \sqrt{k^2 - 4}}{2}$

are the imaginary roots

Therefore $\quad k^2 - 4 < 0$

Factor Theorem – ex. 2



$$k^2 - 4 < 0$$

$$(k - 2)(k + 2) < 0$$

$$-2 < k < 2$$

Answer: $-2 < k < 2$

Binomial Theorem

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Binomial Theorem

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

It is possible to expand the power $(x + y)^n$ into a sum involving terms of the form ax^by^c, where the exponents b and c are nonnegative integers with $b + c = n$

Examples:

$$(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$

$$(a + 2)^3 = a^3 + 3a^2(2) + 3a(2)^2 + (2)^3$$
$$= a^3 + 6a^2 + 12a + 8$$

Binomial Theorem

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

More Examples:

$$(t - 2)^5$$

$$= t^5 + 5t^4(-2) + 10t^3(-2)^2 + 10t^2(-2)^3 + 5t(-2)^4 + (-2)^5$$

$$= t^5 - 10t^4 + 40t^3 - 80t^2 + 80t - 32$$

$$(2x + 3)^3 = (2x)^3 + 3(2x)^2(3) + 3(2x)(3)^2 + (3)^3$$

$$= 8x^3 + 36x^2 + 54x + 27$$

2012

Question 1

(a) Solve the simultaneous equations:

$$a^2 - ab + b^2 = 3$$
$$a + 2b + 1 = 0$$

(b) Find the set of all real values of x for which $\dfrac{2x-5}{x-3} \le \dfrac{5}{2}$.

2012 Sample Paper

Question 4

(a) Solve the simultaneous equations,

$$2x + 8y - 3z = -1$$
$$2x - 3y + 2z = 2$$
$$2x + y + z = 5.$$

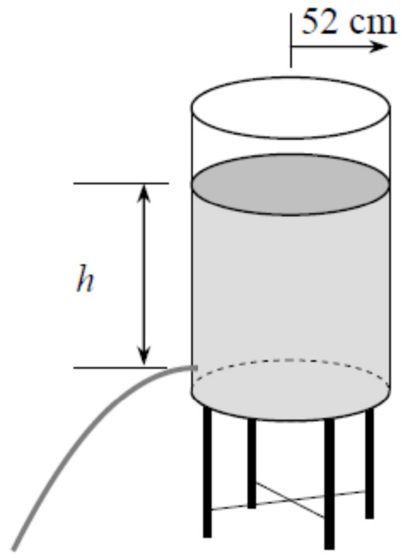
Question 7

An open cylindrical tank of water has a hole near the bottom. The radius of the tank is 52 cm. The hole is a circle of radius 1 cm. The water level gradually drops as water escapes through the hole.

Over a certain 20-minute period, the height of the surface of the water is given by the formula

$$h = \left(10 - \frac{t}{200}\right)^2$$

where h is the height of the surface of the water, in cm, as measured from the centre of the hole,
and t is the time in seconds from a particular instant $t = 0$.



(a) What is the height of the surface at time $t = 0$?

(b) After how many seconds will the height of the surface be 64 cm?

Question 3 **(25 marks)**

(a) The cubic function $f : x \mapsto x^3 + 7x^2 + 17x + 15$ has one integer root and two complex roots. Find all three roots.

(b) Using part **(a)**, or otherwise, solve the equation $(x-2)^3 + 7(x-2)^2 + 17(x-2) + 15 = 0$.

Question 5 **(50 marks)**

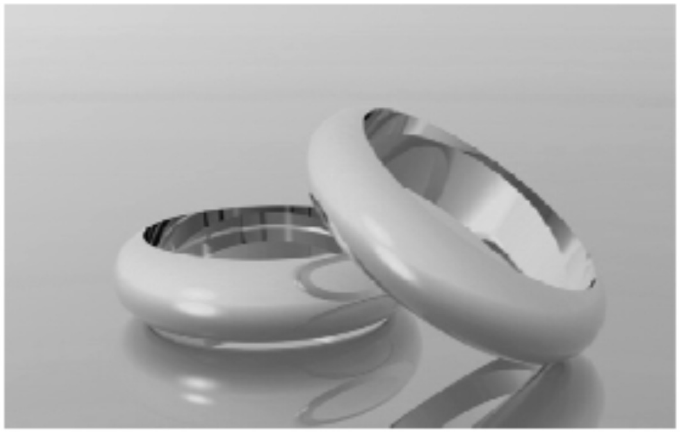
Gold jewellery is made from a gold alloy – that is, a mixture of pure gold and other metals. The purity of the material is measured by its "carat rating", given by the formula

$$c = \frac{24m_g}{m_t}$$

where c = carat rating

m_g = mass of gold in the material

m_t = total mass of the material.



A jeweller is recycling old gold jewellery. He has the following old jewellery in stock:

147 grams of 9-carat gold
85 grams of 18-carat gold.

He can melt down this old jewellery and mix it in various proportions to make new jewellery of different carat values. The value of the old jewellery is equal to the value of its gold content only. Gold is valued at €36 per gram.

(a) What is the total value of the jeweller's stock of old jewellery?

(b) The jeweller wants to make a 15-carat gold pendant weighing 21 grams. He melts down some 9-carat gold and some 18-carat gold to do this. How many grams of each should he use in order to get the 21 grams of 15-carat gold?

(c) The other metals in the gold alloy are copper and silver. The colour of the alloy depends on the ratio of copper to silver. In all of the old jewellery, the amount of silver is equal to the amount of copper. The jeweller has a stock of pure silver that he can add to any mixture. He wants to make an item that:

- weighs 48 grams
- is of 15-carat gold purity
- has twice as much silver as copper.

(i) How many grams of copper will this item contain?

(ii) How many grams of each type of stock (9-carat gold, 18-carat gold, and pure silver) should the jeweller use in order to make this item?

Question 7 **(40 marks)**

(a) Three natural numbers a, b and c, such that $a^2 + b^2 = c^2$, are called a Pythagorean triple.

(i) Let $a = 2n + 1$, $b = 2n^2 + 2n$ and $c = 2n^2 + 2n + 1$.
Pick one natural number n and verify that the corresponding values of a, b and c form a Pythagorean triple.

(ii) Prove that $a = 2n + 1$, $b = 2n^2 + 2n$ and $c = 2n^2 + 2n + 1$, where $n \in \mathbb{N}$, will always form a Pythagorean triple.