chapter 4 Integration

Section 4.1 Introduction to Integration

In general, $\int x^n \, dx = \frac{x^{n+1}}{n+1} + c, n \neq -1$

Example 1

Find (i) $\int (3x^2 + 4x + 5)\, dx$ (ii) $\int (2x - 1)^2 \, dx$.

Example 2

Find (i) $\int \frac{x^3 - 4x}{x}\,dx$ (ii) $\int \left(x^3 + \frac{1}{x^2} + \sqrt{x}\right) dx$ (iii) $\int \sqrt{x}(x + 4)\,dx$

Finding the constant of integration

Each of the examples above contain an arbitrary constant c.
This arbitrary constant is generally called the **constant of integration**.
This constant of integration can be found if further information about the function is given.
This is illustrated in the following example.

Example 3

A curve with equation $y = f(x)$ passes through the point $(2, 0)$.

If $f'(x) = 3x^2 - \frac{1}{x^2}$, find $f(x)$.

Exercise 4.1

1. Find each of the following integrals:

(i) $\int x \, dx$ (ii) $\int x^2 \, dx$ (iii) $\int (3x^2 + 4x) \, dx$

1. Find each of the following integrals:

(iv) $\int -2x^2 \, dx$ (v) $\int 3 \, dx$ (vi) $\int (-x^2 + 3) \, dx$

1. Find each of the following integrals:

(vii) $\int(4x^3 + 6x)\,\mathrm{d}x$ (viii) $\int(2x^2 - 3x - 1)\,\mathrm{d}x$ (ix) $\int 12y^2\,\mathrm{d}y$

6. Find (i) $\displaystyle\int \frac{x^4 - 3x^3 + 4x}{x}\,\mathrm{d}x$ (ii) $\displaystyle\int \frac{3x^3 - x^2 + 6}{x^2}\,\mathrm{d}x$ (iii) $\displaystyle\int \frac{x^2 - 2x + 6}{\sqrt{x}}\,\mathrm{d}x$

7. A curve with equation $y = f(x)$ contains the point $(-1, 4)$.
If $f'(x) = 2x$, find the equation of the curve.

Section 4.2 Integrating exponential and trigonometric functions

Example 1

Find the antiderivative of each of the following:

(i) $\int e^{3x}\,dx$ (ii) $\int (e^{4x} + 6x)\,dx$ (iii) $\int (e^{5x} + 2)\,dx$ (iv) $\int (e^{x} + e^{-x})\,dx$

Example 2

Given $y = 5^x$, use the rules of logarithms to find x in terms of y.

Hence, find (i) $\frac{dx}{dy}$ (ii) $\frac{dy}{dx}$.

Use the result from (ii) to show that $\int 5^x\,dx = \frac{5^x}{\ln 5} + c$.

Example 3

Find (i) $\int \cos 4x \, dx$ (ii) $\int \sin 3x \, dx$.

Example 4

If $y = \sin 3x^2$, find $\frac{dy}{dx}$.

Example 5

Let $h(x) = x \ln x,\ x \in R, x > 0$.

(i) Find $h'(x)$.

(ii) Hence, find $\int \ln x \, dx$.

Exercise 4.2

1. Find the following integrals:

(i) $\int e^{2x} \, dx$ (ii) $\int 3e^{x} \, dx$ (iii) $\int 2e^{4x} \, dx$ (iv) $\int e^{-3x} \, dx$

2. Integrate each of the following:

(i) $\int (e^{3x} + 4)\,dx$ (ii) $\int 4e^{\frac{1}{2}x}\,dx$ (iii) $\int \left(e^{4x} + \frac{1}{e^{4x}}\right) dx$

5. Integrate each of the following:

(i) $\int 3\cos 6x\,dx$ (ii) $\int (\cos 2x - \sin 5x)\,dx$ (iii) $\int 3\cos(-9x)\,dx$

14. Let $f(x) = 2x\,e^x$.

(i) Find $f'(x)$.

(ii) Hence, find $\int 2x\,e^x\,dx$.

15. Given $f(x) = x\sin x$, find $f'(x)$.

Hence, find $\int x\cos x\,dx$.

Section 4.3 Applications of integration

Example 1

A body moves in a straight line.
At time t seconds, its acceleration is given by $a = 6t + 1$.
When $t = 0$, the velocity of the body is 2 m/sec and its displacement from a fixed point O is 1 metre.

(i) Find expressions for v and s in terms of t.

(ii) Find the velocity of the body after 4 seconds.

3. The acceleration of a body is given by $a = 6t - 12$.

(i) Find the velocity v in terms of t, given that $v = 9$ when $t = 0$.
(ii) Find the displacement s in terms of t, given that $s = 6$ when $t = 0$.
(iii) Find the values of t when the body is at rest.

6. $\frac{dN}{dt} = 4e^t + 10$ represents the rate at which a colony of bacteria increases, where N is the number of bacteria and t is measured in hours.

(i) Find an expression for N in terms of t.
(ii) If there were 10 bacteria in the colony initially, find the number in the colony after 5 hours, correct to the nearest whole number.

Section 4.4 Definite integrals

Example 1

Evaluate (i) $\int_0^2 3x^2 \, dx$ (ii) $\int_2^4 (x^2 - x + 3) \, dx$ (iii) $\int_4^9 \frac{1}{\sqrt{x}} \, dx$

Example 2

Evaluate (i) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos 2x \, dx$ (ii) $\int_{2}^{5} 4e^{x} \, dx$ (iii) $\int_{0}^{2} 9^{x} \, dx$

1. $\int_{1}^{2} 6x \, dx$ **2.** $\int_{1}^{3} (3x^2 - 2x) \, dx$ **3.** $\int_{1}^{4} (3x^2 - 4) \, dx$

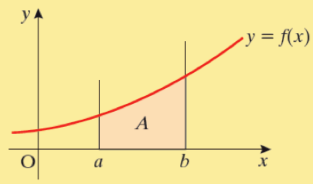
26. Find $\frac{\mathrm{d}}{\mathrm{d}x}(x \sin 3x)$.

Hence, evaluate $\int_0^{\frac{\pi}{6}} 3x \cos 3x \, \mathrm{d}x$.

Section 4.5 Finding areas by integration

The area, A, of the region between the curve $y = f(x)$ and the x-axis between the lines $x = a$ and $x = b$ is given by

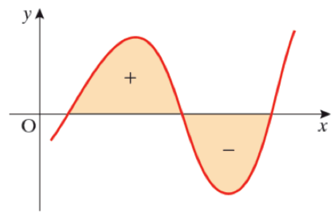
$$A = \int_a^b f(x)\,dx$$



When using $\int_a^b y\,dx$ to find the area between a curve and the x-axis, the areas of the regions above and below the x-axis must be found separately.

If $b > a$, the value of $\int_a^b y\,dx$ will be positive if the area enclosed is above the x-axis, and negative if the area is below the x-axis.

If an area is -16, we take the absolute value, 16, to be the area.



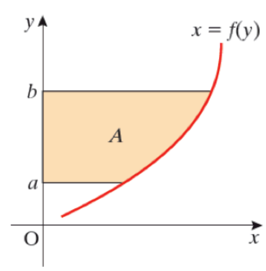
Area between a curve and the *y*-axis

If we require the area between a curve and the y-axis, the function must be written in the form $x = f(y)$.

The area of the shaded region between the curve and the y-axis between the lines $y = b$ and $y = a$ is given by:

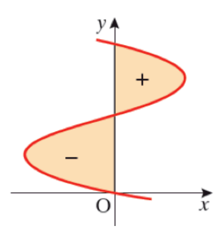
Area between a curve and the y-axis

Area $A = \int_a^b x\,dy$



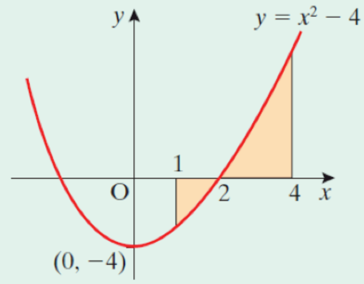
If the region is to the right of the y-axis, the area is positive; if the region is to the left of the y-axis, the area is negative.

Areas to the right and to the left of the y-axis must be found separately and then added.



Example 1

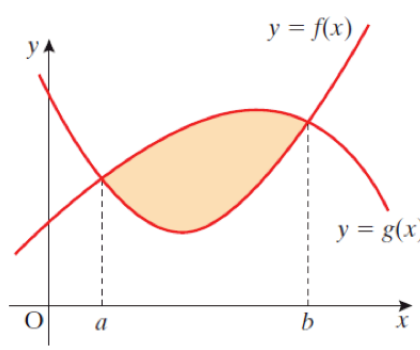
Find the area of the shaded region shown in the given diagram.



Area between two curves

The given figure shows two curves $y = f(x)$ and $y = g(x)$ intersecting at the points where $x = a$ and $x = b$.

The shaded area $= \int_a^b g(x)\,dx - \int_a^b f(x)\,dx$



Example 2

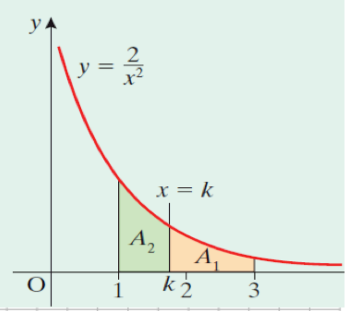
Find the area of the region bounded by the curve $y = -x^2 + 5x - 4$ and the line $y = x - 1$.

Example 3

The diagram on the right shows a sketch of the function $y = \frac{2}{x^2}$.

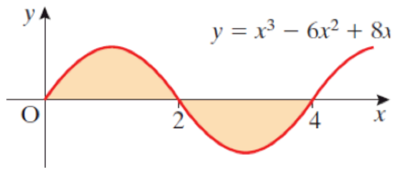
The shaded region represents the area bounded by the curve and the x-axis between the lines $x = 3$ and $x = 1$.

If the line $x = k$ divides this area into two equal portions, find the value of k.



Find the area of the shaded region in numbers (1–8):

5.



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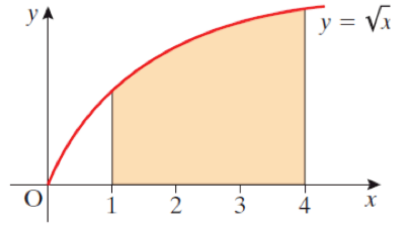
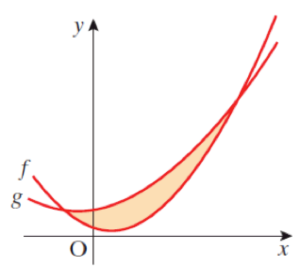


23. The functions f and g are defined for $x \in R$ as,

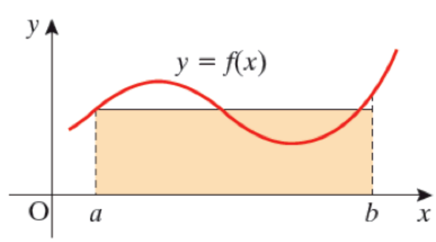
$f(x) = 2x^2 - 3x + 2$ and
$g(x) = x^2 + x + 7$.

(i) Find the coordinates of the two points where the curves $y = f(x)$ and $y = g(x)$ intersect.

(ii) Find the area of the region enclosed between the two curves.



Section 4.6 Average value of a function

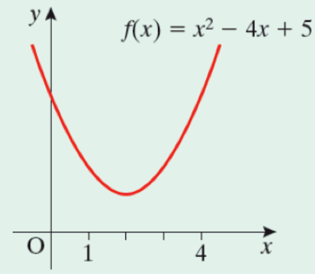


The average value of a function $f(x)$ over the interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x.$$

Example 1

The graph of the function, $f(x) = x^2 - 4x + 5$ is shown. Find the average value of the function for $1 \leqslant x \leqslant 4$.



Example 2

A body starts from rest and moves in a straight line.
After t seconds its velocity (v) is given by $v = 2t - 4, t \geqslant 0$.

(i) By completing the table on the right, find the average velocity over the first 3 seconds.

$t =$	0	1	2	3
$v =$				

(ii) Use integration to test the accuracy of your answer.

Example 3

The average value of the function $f(x) = 2x + 3$ for $1 \leqslant x \leqslant k$ is 11. Find the value of k.

4. Find the average value of the function $f(x) = x^2 + 4$ for $-2 \leqslant x \leqslant 3$.

13. The tension T newtons in a particular spring depends on the extension x metres of the spring from its natural length in accordance with the rule $T = 30x$.
Find the average tension in the spring as x increases from 0.1 m to 0.2 m.

Syllabus

- recognise integration as the reverse process of differentiation
- use integration to find the average value of a function over an interval

- integrate sums, differences and constant multiples of functions of the form
  - x^a, where $a \in \mathbf{Q}$
  - a^x, where $a \in \mathbf{R}$
  - Sin ax, where $a \in \mathbf{R}$
  - Cos ax, where $a \in \mathbf{R}$
- determine areas of plane regions bounded by polynomial and exponential curves

Maths Tables

Integration

Constants of integration omitted.

$f(x)$	$\int f(x)dx$
$x^n, (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\|x\|$
e^x	e^x
e^{ax}	$\frac{1}{a}e^{ax}$
$a^x \quad (a > 0)$	$\frac{a^x}{\ln a}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\tan x$	$\ln\|\sec x\|$
$\frac{1}{\sqrt{a^2 - x^2}} \quad (a > 0)$	$\sin^{-1}\frac{x}{a}$
$\frac{1}{x^2 + a^2} \quad (a > 0)$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$