chapter 4 **Integration**

Section 4.1 Introduction to Integration

" Increase power by 1 divide by new power +c "

DIFFERENTIATE

$\boxed{\text{FUNCTION}}$ $\qquad\qquad$ $\boxed{\text{DERIVATIVE}}$

$3x^2 + 2 \qquad\qquad 6x$

$3x^2 + C$

ANTI-DIFFERENTIATE
(INTEGRATE)

In general, $\int x^n \, dx = \dfrac{x^{n+1}}{n+1} + c, n \neq -1$

Example 1

Find (i) $\int (3x^2 + 4x + 5) \, dx$ (ii) $\int (2x - 1)^2 \, dx$.

(i) $\int (3x^2 + 4x + 5x^0) \, dx = \int 3x^2 dx + \int 4x dx + \int 5 dx$

$= 3\int x^2 dx + 4\int x dx + 5\int dx$

$= \dfrac{\cancel{3}x^3}{\cancel{3}} + \dfrac{\cancel{4}^{2}x^2}{\cancel{2}} + 5x + C$

$= x^3 + 2x^2 + 5x + C$

(ii) $\int (2x-1)^2 dx = \int (4x^2 - 4x + 1) dx$

$= \dfrac{4x^3}{3} - \dfrac{4x^2}{2} + 1x + c$

$= \dfrac{4}{3}x^3 - 2x^2 + x + c$

Example 2

Find (i) $\int \frac{x^3 - 4x}{x}\,dx$ (ii) $\int \left(x^3 + \frac{1}{x^2} + \sqrt{x}\right)dx$ (iii) $\int \sqrt{x}(x+4)\,dx$

Rewrite as polynomial with powers of x.

"increase the power by 1, divide by new power +c"

$$\int \left(x^3 + \frac{1}{x^2} + \sqrt{x}\right)dx = \int \left(x^3 + x^{-2} + x^{\frac{1}{2}}\right)dx$$

$$= \frac{x^4}{4} + \frac{x^{-1}}{-1} + \frac{x^{3/2}}{3/2} + c$$

$$= \frac{x^4}{4} - \frac{1}{x} + \frac{2}{3}x^{3/2} + c$$

Finding the constant of integration

Each of the examples above contain an arbitrary constant c.
This arbitrary constant is generally called the **constant of integration**.
This constant of integration can be found if further information about the function is given.
This is illustrated in the following example.

Example 3

A curve with equation $y = f(x)$ passes through the point $(2, 0)$.

If $f'(x) = 3x^2 - \frac{1}{x^2}$, find $f(x)$.

$$f(x) = \int \left(3x^2 - \frac{1}{x^2}\right)dx$$

$$= \int \left(3x^2 - x^{-2}\right)dx$$

$$= \frac{3x^3}{3} - \frac{x^{-1}}{-1} + c$$

$$f(x) = x^3 + x^{-1} + c$$

Given $f(2) = 0$

$$f(2) = (2)^3 + \frac{1}{(2)} + c = 0$$

$$\Rightarrow 8 + \tfrac{1}{2} + c = 0 \quad \Rightarrow \quad c = -\tfrac{17}{2}$$

$$f(x) = x^3 + x^{-1} - 17/2$$

Exercise 4.1

1. Find each of the following integrals:

(i) $\int x\,dx$ (ii) $\int x^2\,dx$ (iii) $\int (3x^2 + 4x)\,dx$

Increase the power by 1 divide by the new power +c

$$\int x\,dx = \frac{x^2}{2} + c$$

$$\int x^2\,dx = \frac{x^3}{3} + c$$

$$\int (3x^2 + 4x)\,dx = \frac{3x^3}{3} + \frac{4x^2}{2} + c$$
$$= x^3 + 2x^2 + c$$

1. Find each of the following integrals:

(iv) $\int -2x^2\,dx$ (v) $\int 3\,dx$ (vi) $\int (-x^2 + 3)\,dx$

Increase the power by 1 divide by the new power +c

$$\int -2x^2\,dx = -\frac{2x^3}{3} + c$$

$$\int 3\,dx = 3x + c$$

$$\int (-x^2 + 3)\,dx = -\frac{x^3}{3} + 3x + c$$

1. Find each of the following integrals:

(vii) $\int(4x^3 + 6x)\,dx$ (viii) $\int(2x^2 - 3x - 1)\,dx$ (ix) $\int 12y^2\,dy$

Increase the power by 1 divide by the new power +c

$$\int(4x^3+6x)\,dx = \frac{4x^4}{4} + \frac{6x^2}{2} + c$$
$$= x^4 + 3x^2 + c$$

$$\int(2x^2-3x-1)\,dx = \frac{2x^3}{3} - \frac{3x^2}{2} - 1x + c$$

$$\int 12y^2\,dy = \frac{12y^3}{3} + c$$
$$= 4y^3 + c$$

Section 4.2 Integrating exponential and trigonometric functions

INTEGRATE

$$\cos ax \longrightarrow \frac{1}{a}\sin ax$$

$$\sin ax \longrightarrow -\frac{1}{a}\cos ax$$

$$e^{ax} \longrightarrow \frac{1}{a}e^{ax}$$

$$e^{x} \longrightarrow e^{x}$$

Example 1

Find the antiderivative of each of the following:

(i) $\int e^{3x}\,dx$ (ii) $\int (e^{4x} + 6x)\,dx$ (iii) $\int (e^{5x} + 2)\,dx$ (iv) $\int (e^{x} + e^{-x})\,dx$

INTEGRATION

$e^{ax} \rightarrow \frac{1}{a} e^{ax}$

$e^{x} \rightarrow e^{x}$

(i) $\int e^{3x}\,dx = \frac{1}{3} e^{3x} + c$

(ii) $\int (e^{4x} + 6x)\,dx = \frac{1}{4} e^{4x} + \frac{6x^2}{2} + c$

$= \frac{1}{4} e^{4x} + 3x^2 + c$

(iii) $\int (e^{5x} + 2)\,dx = \frac{1}{5} e^{5x} + 2x + c$

(iv) $\int (e^{x} + e^{-x})\,dx = e^{x} + \frac{1}{-1} e^{-x} + c$

$= e^{x} - e^{-x} + c$

Example 3

Find (i) $\int \cos 4x\,dx$ (ii) $\int \sin 3x\,dx$.

Remember

$y = \sin 2x$ $\xrightarrow{\text{DIFFERENTIATE}}$ $\frac{dy}{dx} = 2\cos 2x$

$2\cos 2x \xrightarrow{\text{INTEGRATE}} \sin 2x + c$

Rule: INTEGRATION

$\cos ax \rightarrow \frac{1}{a} \sin ax$

$\sin ax \rightarrow -\frac{1}{a} \cos ax$

*learn these results!

(i) $\int \cos 4x\,dx = \frac{1}{4} \sin 4x + c$

(ii) $\int \sin 3x\,dx = -\frac{1}{3} \cos 3x + c$

Example 5

Let $h(x) = x \ln x,\ x \in R, x > 0$.

(i) Find $h'(x)$.

(ii) Hence, find $\int \ln x \, dx$.

*

Differentiate (i)

PRODUCT RULE

$h(x) = x \ln x$

$h'(x) = x\left(\frac{1}{x}\right) + 1(\ln x)$

$h'(x) = 1 + \ln x$

$\boxed{\text{FUNCTION}}$ $+c$

DERIVATIVE, INTEGRATE

$\boxed{\text{DERIVATIVE}}$

Integrate (ii)

$\int (1 + \ln x)\, dx = x \ln x + C$

$x + \int \ln x \, dx = x \ln x + C$

$\int \ln x \, dx = x \ln x - x + c$

5. Integrate each of the following:

(i) $\int 3 \cos 6x \, dx$ (ii) $\int (\cos 2x - \sin 5x)\, dx$ (iii) $\int 3 \cos(-9x)\, dx$

(iii)

Integrate

$\cos ax \rightarrow \frac{1}{a} \sin ax$

$\int 3 \cos(-9x)\, dx$

$= 3\left(\frac{1}{-9}\right) \sin(-9x) + C$

$= -\frac{1}{3} \sin(-9x) + c$

14. Let $f(x) = 2x\,e^x$.

(i) Find $f'(x)$.

(ii) Hence, find $\int 2x\,e^x\,dx$.

Differentiate (i)

PRODUCT RULE

DIFFERENTIATE
$e^x \to e^x$

$f(x) = 2xe^x$

$f'(x) = 2x(e^x) + 2(e^x)$

Integrate (ii)

integrate
$e^x \to e^x$

$\int (2xe^x + 2e^x)\,dx = 2xe^x + C$

$\int 2xe^x\,dx + 2e^x = 2xe^x + C$

$\int 2xe^x\,dx = 2xe^x - 2e^x + C$

15. Given $f(x) = x\sin x$, find $f'(x)$.

Hence, find $\int x\cos x\,dx$.

Step 1 : Differentiate

PRODUCT RULE

DIFFERENTIATE
$\sin x \to \cos x$

$f(x) = x\sin x$ $\boxed{\text{FUNCTION}}$

$f'(x) = x(\cos x) + 1(\sin x)$

$f'(x) = x\cos x + \sin x$ $\boxed{\text{DERIVATIVE}}$

Step 2 : Integrate

"Integral of derivative = function + C"

INTEGRATE
$\sin x \to -\cos x$

INTEGRAL OF DERIVATIVE = FUNCTION + C

$\int (x\cos x + \sin x)\,dx = x\sin x + C$

$\int x\cos x\,dx - \cos x = x\sin x + C$

$\int x\cos x\,dx = x\sin x + \cos x + C$

Section 4.3 Applications of integration

DIFFERENTIATE ↓

s DISTANCE / DISPLACEMENT $\int v\,dt$

$\frac{ds}{dt} = v$ VELOCITY $\int a\,dt$

$\frac{dv}{dt}$ ACCELERATION a

INTEGRATE ↑

Example 1

A body moves in a straight line.
At time t seconds, its acceleration is given by $a = 6t + 1$.
When $t = 0$, the velocity of the body is 2 m/sec and its displacement from a fixed point O is 1 metre.

(i) Find expressions for v and s in terms of t.

(ii) Find the velocity of the body after 4 seconds.

Velocity?

$v = \int a\,dt$

$a = 6t + 1$

$v = \int (6t+1)\,dt$

$= \frac{6t^2}{2} + t + c$

$v = 3t^2 + t + c$

When $t=0$, $v = 2$m/s

$2 = 3(0)^2 + (0) + c \Rightarrow \boxed{2 = c}$

$v = 3t^2 + t + 2$

$t=4$ $\Rightarrow v = 3(4)^2 + 4 + 2 = 54$m/s

$s = \int v\,dt$

$s = \int (3t^2 + t + 2)\,dt = \frac{\cancel{3}t^3}{\cancel{3}} + \frac{t^2}{2} + 2t + c$

$s = t^3 + t^2/2 + 2t + c$

When $t=0$, $s = 1$m

$1 = 0^3 + 0^2/2 + 2(0) + c \Rightarrow \boxed{c = 1}$

$s = t^3 + \frac{1}{2}t^2 + 2t + 1$

3. The acceleration of a body is given by $a = 6t - 12$.

(i) Find the velocity v in terms of t, given that $v = 9$ when $t = 0$.
(ii) Find the displacement s in terms of t, given that $s = 6$ when $t = 0$.
(iii) Find the values of t when the body is at rest.

$a = 6t - 12$

$v = \int a\,dt$ (i) $v = \int (6t - 12)\,dt = \frac{\cancel{6}3t^2}{\cancel{2}} - 12t + c$

When $v = 9$, $t = 0$: $9 = 0^2 - 12(0) + c \Rightarrow c = 9$

$v = 3t^2 - 12t + 9$

$s = \int v\,dt$ (ii) $s = \int (3t^2 - 12t + 9)\,dt = \frac{\cancel{3}t^3}{\cancel{3}} - \frac{\cancel{12}6t^2}{\cancel{2}} + 9t + c$

When $s = 6$, $t = 0$: $6 = 0^3 - 6(0)^2 + 9(0) + c \Rightarrow c = 6$

$s = t^3 - 6t^2 + 9t + 6$

"Body at Rest" $\Rightarrow v = 0$, $t = ?$ (iii) $v = 3t^2 - 12t + 9 = 0$

$\Rightarrow t^2 - 4t + 3 = 0$

$(t-3)(t-1) = 0$

$t = 3$ s or $t = 1$ s.

Section 4.4 Definite integrals

Example 1

Evaluate (i) $\int_0^2 3x^2\,dx$ (ii) $\int_2^4 (x^2 - x + 3)\,dx$ (iii) $\int_4^9 \frac{1}{\sqrt{x}}\,dx$

limits

(i)

Definite Integral

$$\int_0^2 3x^2\,dx$$
$$= \left[\frac{\cancel{3}x^3}{\cancel{3}} + c\right]_0^2$$
$$= \left((2)^3 + c\right) - \left((0)^3 + c\right)$$
$$= 8 + \cancel{c} - 0 - \cancel{c}$$
$$= 8$$

notice no c in answer

(iii)

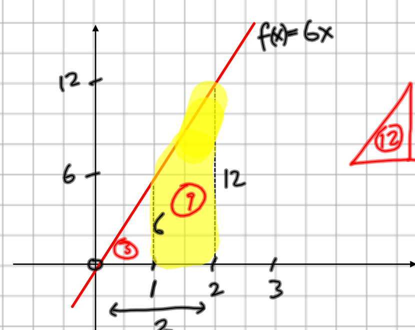
"increase power by 1 and divide by the new power"

$$\int_4^9 \frac{1}{\sqrt{x}}\,dx = \int_4^9 x^{-\frac{1}{2}}\,dx$$
$$= \left[\frac{x^{\frac{1}{2}}}{\left(\frac{1}{2}\right)}\right]_4^9 = \left[2\sqrt{x}\right]_4^9$$
$$= (2\sqrt{9}) - (2\sqrt{4}) = 6 - 4 = 2$$

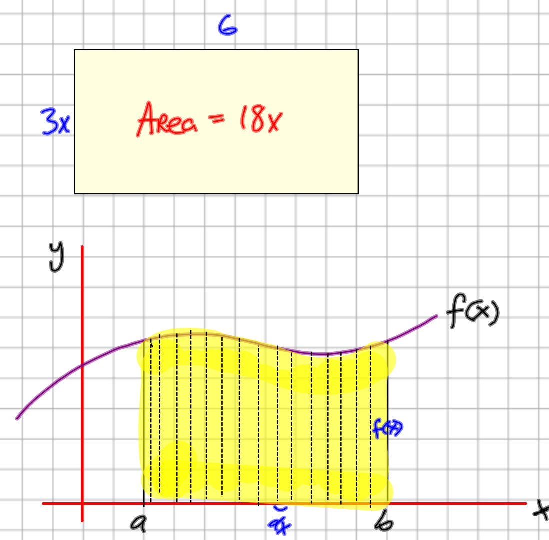
1. $\int_1^2 6x\,dx$ **2.** $\int_1^3 (3x^2 - 2x)\,dx$ **3.** $\int_1^4 (3x^2 - 4)\,dx$

Definite integral

$$\int_1^2 6x\,dx = \left[\frac{3\cancel{6}x^2}{\cancel{2}}\right]_1^2$$
$$= \left(3(2)^2\right) - \left(3(1)^2\right)$$
$$= 12 - 3 = 9$$



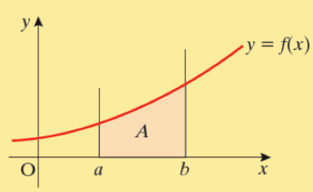
Section 4.5 Finding areas by integration



$$A = \int_a^b f(x)\,dx$$

The area, A, of the region between the curve $y = f(x)$ and the x-axis between the lines $x = a$ and $x = b$ is given by

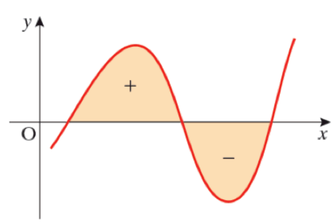
$$A = \int_a^b f(x)\,\mathrm{d}x$$



When using $\int_a^b y\,\mathrm{d}x$ to find the area between a curve and the x-axis, the areas of the regions above and below the x-axis must be found separately.

If $b > a$, the value of $\int_a^b y\,\mathrm{d}x$ will be positive if the area enclosed is above the x-axis, and negative if the area is below the x-axis.

If an area is -16, we take the absolute value, 16, to be the area.



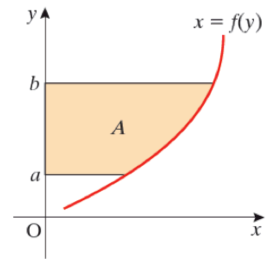
Area between a curve and the y-axis

If we require the area between a curve and the y-axis, the function must be written in the form $x = f(y)$.

The area of the shaded region between the curve and the y-axis between the lines $y = b$ and $y = a$ is given by:

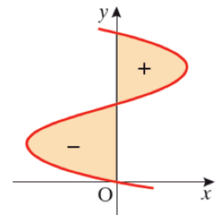
Area between a curve and the y-axis

$$\text{Area } A = \int_a^b x\,dy$$



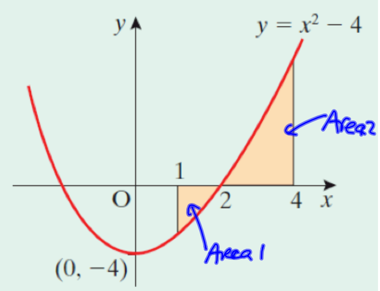
If the region is to the right of the y-axis, the area is positive; if the region is to the left of the y-axis, the area is negative.

Areas to the right and to the left of the y-axis must be found separately and then added.



Example 1

Find the area of the shaded region shown in the given diagram.



$$\boxed{A = \int_a^b f(x).dx}$$

Area 1 + Area 2

$$\text{Area} = \left|\int_1^2 (x^2-4)\,dx\right| + \int_2^4 (x^2-4)\,dx$$

$$= \left|\left[\frac{x^3}{3} - 4x\right]_1^2\right| + \left[\frac{x^3}{3} - 4x\right]_2^4$$

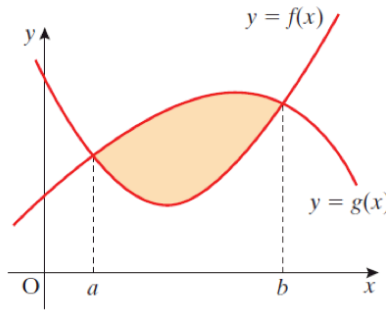
$$= \left|\left(\frac{2^3}{3} - 4(2)\right) - \left(\frac{1^3}{3} - 4(1)\right)\right| + \left[\left(\frac{4^3}{3} - 4(4)\right) - \left(\frac{2^3}{3} - 4(2)\right)\right]$$

$$= 21$$

Area between two curves

The given figure shows two curves $y = f(x)$ and $y = g(x)$ intersecting at the points where $x = a$ and $x = b$.

The shaded area $= \int_a^b g(x)\,dx - \int_a^b f(x)\,dx$



Example 2

Find the area of the region bounded by the curve $y = -x^2 + 5x - 4$ and the line $y = x - 1$.

limits? Intersection?

limits are $x = 1$ & 3

Area between Curves

$$\boxed{A = \int_a^b (f(x) - g(x))\,dx}$$

$$x - 1 = -x^2 + 5x - 4$$
$$x^2 - 4x + 3 = 0$$
$$(x-3)(x-1) = 0$$
$$x = 3, \quad x = 1$$

$$A = \int_1^3 [(-x^2 + 5x - 4) - (x-1)]\,dx$$
$$= \int_1^3 (-x^2 + 4x - 3)\,dx$$

Integrate

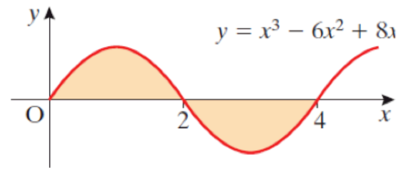
$$= \left[-\frac{x^3}{3} + \frac{\overset{2}{\cancel{4}}x^2}{\cancel{2}} - 3x\right]_1^3$$

evaluate

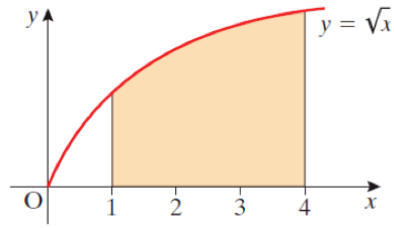
$$= \left(-\frac{(3)^3}{3} + 2(3)^2 - 3(3)\right) - \left(-\frac{(1)^3}{3} + 2(1)^2 - 3(1)\right)$$
$$= \frac{4}{3}$$

Find the area of the shaded region in numbers (1–8):

5.



6.



6.

$A = \int_a^b f(x)\,dx$

Integrate and evaluate

$f(x) = \sqrt{x} = x^{1/2}$

$A = \int_1^4 x^{1/2}\,dx = \left[\frac{x^{3/2}}{3/2}\right]_1^4$

$= \left[\frac{2}{3}x^{3/2}\right]_1^4$

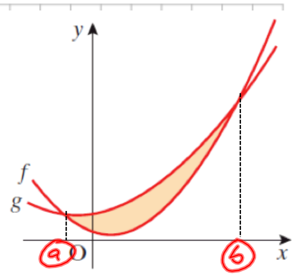
$= \left(\frac{2}{3}(4)^{3/2}\right) - \left(\frac{2}{3}(1)^{3/2}\right)$

$= \frac{14}{3}$

23. The functions f and g are defined for $x \in R$ as,

$f(x) = 2x^2 - 3x + 2$ and
$g(x) = x^2 + x + 7$.

(i) Find the coordinates of the two points where the curves $y = f(x)$ and $y = g(x)$ intersect.
(ii) Find the area of the region enclosed between the two curves.



Limits? Intersection?

$2x^2 - 3x + 2 = x^2 + x + 7$
$x^2 - 4x - 5 = 0$
$(x - 5)(x + 1) = 0$

LIMITS: $x = 5$, $x = -1$

Area between curves

$A = \int_a^b (f(x) - g(x))\,dx$

$A = \int_{-1}^{5}\left[(2x^2 - 3x + 2) - (x^2 + x + 7)\right]dx$

$= \int_{-1}^{5} (x^2 - 4x - 5)\,dx$

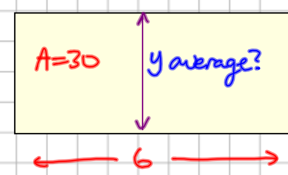
$= \left[\frac{x^3}{3} - {}^{2}\cancel{4}\frac{x^2}{\cancel{2}} - 5x\right]_{-1}^{5}$

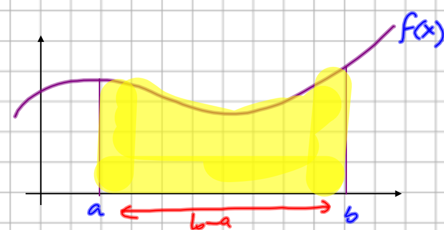
$= \left[\frac{(5)^3}{3} - 2(5)^2 - 5(5)\right] - \left[\frac{(-1)^3}{3} - 2(-1)^2 - 5(-1)\right]$

$= 36$ units^2

Section 4.6 Average value of a function

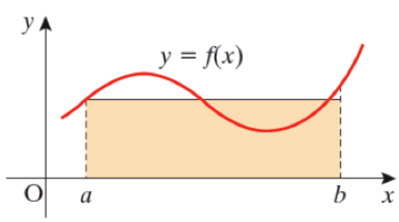
Average y-Value





$A = \int_a^b f(x)\,dx$

$$\boxed{\text{Average Value} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

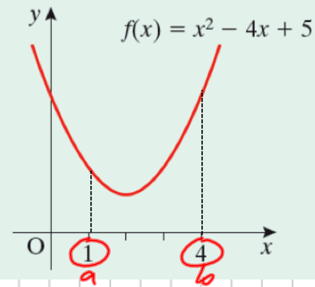


The average value of a function $f(x)$ over the interval $[a, b]$ is

$$\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x.$$

Example 1

The graph of the function, $f(x) = x^2 - 4x + 5$ is shown. Find the average value of the function for $1 \leqslant x \leqslant 4$.



$$\text{Average Value} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

$b-a = 4-1 = 3$

integrate & evaluate

$$\text{Average Value} = \frac{1}{3}\int_1^4 (x^2-4x+5)\,dx$$

$$= \frac{1}{3}\left[\frac{x^3}{3} - \frac{\cancel{4}^{2}x^2}{\cancel{2}} + 5x\right]_1^4$$

$$= \frac{1}{3}\left[\left(\frac{4^3}{3} - 2(4)^2 + 5(4)\right) - \left(\frac{1^3}{3} - 2(1) + 5(1)\right)\right]$$

$$= 2$$

Example 3

The average value of the function $f(x) = 2x + 3$ for $1 \leqslant x \leqslant k$ is 11. Find the value of k.

$$\text{Average Value} = \frac{1}{b-a}\int f(x)\,dx$$

$b-a = k-1$

$$11 = \frac{1}{k-1}\int_1^k (2x+3)\,dx$$

$$11(k-1) = \left[\frac{\cancel{2}x^2}{\cancel{2}} + 3x\right]_1^k$$

$$11k - 11 = (k^2+3k) - (1^2 + 3(1))$$

$$11k - 11 = k^2 + 3k + 4$$

$$k^2 - 8k + 7 = 0$$

$$(k-7)(k-1) = 0$$

$k = 7$ or $k = 1$ reject

$\Rightarrow k = 7$

Syllabus

- recognise integration as the reverse process of differentiation
- use integration to find the average value of a function over an interval

- integrate sums, differences and constant multiples of functions of the form
  - x^a, where $a \in \mathbf{Q}$
  - a^x, where $a \in \mathbf{R}$
  - Sin ax, where $a \in \mathbf{R}$
  - Cos ax, where $a \in \mathbf{R}$
- determine areas of plane regions bounded by polynomial and exponential curves

Maths Tables

Integration

Constants of integration omitted.

$f(x)$	$\int f(x)dx$
$x^n,\ (n \neq -1)$	$\frac{x^{n+1}}{n+1}$
$\frac{1}{x}$	$\ln\lvert x\rvert$
e^x	e^x
e^{ax}	$\frac{1}{a}e^{ax}$
$a^x\ \ (a>0)$	$\frac{a^x}{\ln a}$
$\cos x$	$\sin x$
$\sin x$	$-\cos x$
$\tan x$	$\ln\lvert\sec x\rvert$
$\frac{1}{\sqrt{a^2-x^2}}\ \ (a>0)$	$\sin^{-1}\frac{x}{a}$
$\frac{1}{x^2+a^2}\ \ (a>0)$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$

learn $\begin{cases} \cos ax \rightarrow \frac{1}{a}\sin ax \\ \sin ax \rightarrow -\frac{1}{a}\cos ax \end{cases}$