Chapter 4: Sequences–Series–Patterns

Exercise 4.1

Q**1**. (i) $6, 12, 18, 24, 30, 36, 42$

(ii) $7, 12, 17, 22, 27, 32, 37$

(iii) $4.7, 5.9, 7.1, 8.3, 9.5, 10.7, 11.9$

(iv) $2, -1, -4, -7, -10, -13, -16$

(v) $2, 3, 6, 11, 18, 27, 38, 51, 66$

(vi) $78, 70, 62, 54, 46, 38, 30$

(vii) $10, 5, 0, -5, -10, -15, -20, -25$

(viii) $-64, -55, -46, -37, -28, -19, -10$

(ix) $2, 6, 18, 54, 162, 486$

(x) $2, 6, 12, 20, 30, 42, 56$

(xi) $\frac{3}{4}, \frac{1}{4}, -\frac{1}{4}, -\frac{3}{4}, -\frac{5}{4}, -\frac{7}{4}$

(xii) $1, 2, 4, 7, 11, 16, 22, 29$

(xiii) $0, 3, 8, 15, 24, 35, 48, 63$

(xiv) $3, -6, 12, -24, 48, -96, 192$

(xv) $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}, \frac{1}{42}, \frac{1}{56}$

Q**2**. (i) $T_n = 4n - 2$

$$\Rightarrow \left.\begin{aligned} T_1 &= 4(1) - 2 = 2 \\ T_2 &= 4(2) - 2 = 6 \\ T_3 &= 4(3) - 2 = 10 \\ T_4 &= 4(4) - 2 = 14 \end{aligned}\right\} \quad 2, 6, 10, 14$$

(ii) $T_n = (n+1)^2$

$$\Rightarrow \left.\begin{aligned} T_1 &= (1+1)^2 = 4 \\ T_2 &= (2+1)^2 = 9 \\ T_3 &= (3+1)^2 = 16 \\ T_4 &= (4+1)^2 = 25 \end{aligned}\right\} \quad 4, 9, 16, 25$$

(iii) $T_n = n^2 - 2n$

$$\Rightarrow \left.\begin{aligned} T_1 &= 1^2 - 2(1) = -1 \\ T_2 &= 2^2 - 2(2) = 0 \\ T_3 &= 3^2 - 2(3) = 3 \\ T_4 &= 4^2 - 2(4) = 8 \end{aligned}\right\} \quad -1, 0, 3, 8$$

(iv) $T_n = (n+3)(n+1)$

$$\left.\begin{aligned} \Rightarrow T_1 &= (1+3)(1+1) = 8 \\ T_2 &= (2+3)(2+1) = 15 \\ T_3 &= (3+3)(3+1) = 24 \\ T_4 &= (4+3)(4+1) = 35 \end{aligned}\right\} \quad 8,\ 15,\ 24,\ 35$$

(v) $T_n = n^3 - 1$

$$\left.\begin{aligned} \Rightarrow T_1 &= (1)^3 - 1 = 0 \\ T_2 &= (2)^3 - 1 = 7 \\ T_3 &= (3)^3 - 1 = 26 \\ T_4 &= (4)^3 - 1 = 63 \end{aligned}\right\} \quad 0,\ 7,\ 26,\ 63$$

(vi) $T_n = \dfrac{n}{n+2}$

$$\left.\begin{aligned} \Rightarrow T_1 &= \frac{1}{1+2} = \frac{1}{3} \\ T_2 &= \frac{2}{2+2} = \frac{2}{4} = \frac{1}{2} \\ T_3 &= \frac{3}{3+2} = \frac{3}{5} \\ T_4 &= \frac{4}{4+2} = \frac{4}{6} = \frac{2}{3} \end{aligned}\right\} \quad \frac{1}{3},\ \frac{1}{2},\ \frac{3}{5},\ \frac{2}{3}$$

(vii) $T_n = 2^n$

$$\left.\begin{aligned} \Rightarrow T_1 &= 2^1 = 2 \\ T_2 &= 2^2 = 4 \\ T_3 &= 2^3 = 8 \\ T_4 &= 2^4 = 16 \end{aligned}\right\} \quad 2,\ 4,\ 8,\ 16$$

(viii) $T_n = (-3)^n$

$$\left.\begin{aligned} \Rightarrow T_1 &= (-3)^1 = -3 \\ T_2 &= (-3)^2 = 9 \\ T_3 &= (-3)^3 = -27 \\ T_4 &= (-3)^4 = 81 \end{aligned}\right\} \quad -3,\ 9,\ -27,\ 81$$

(ix) $T_n = n.2^n$

$$\left.\begin{aligned} \Rightarrow T_1 &= 1.2^1 = 2 \\ T_2 &= 2.2^2 = 8 \\ T_3 &= 3.2^3 = 24 \\ T_4 &= 4.2^4 = 64 \end{aligned}\right\} \quad 2,\ 8,\ 24,\ 64$$

Q**3**. (i) $5\,\text{cm},\ 9\,\text{cm},\ 13\,\text{cm},\ 17\,\text{cm},$ etc.

(ii) $T_1 = 5$

$T_2 = 5 + 4$

$T_3 = 5 + 2(4)$

$T_4 = 5 + 3(4)$ etc.

$\Rightarrow T_n = 5 + (n-1)4 = 5 + 4n - 4 = 4n + 1$

$\Rightarrow T_5 = 4(5) + 1 = 21\,\text{cm}.$

Q**4**. (i) $w_1 = 1\,\text{km}$

$w_2 = (1+1)\,\text{km} \quad = 2\,\text{km}$

$w_3 = (2+2)\,\text{km} \ = 4\,\text{km}$

$w_4 = (4+3)\,\text{km} \ = 7\,\text{km}$

$w_5 = (7+4)\,\text{km} \ = 11\,\text{km}$

$w_6 = (11+5)\,\text{km} = 16\,\text{km}$

(ii) $w_7 = (16+6)\,\text{km} \quad = 22\,\text{km}$

$w_8 = (22+7)\,\text{km} \quad = 29\,\text{km}$

In week 8, she ran $29\,\text{km}$.

Q**5**. $T_n = 4n - 3$

$\Rightarrow T_1 = 4(1) - 3 \quad = 1$

$T_5 = 4(5) - 3 \quad = 17$

$T_{10} = 4(10) - 3 = 37.$

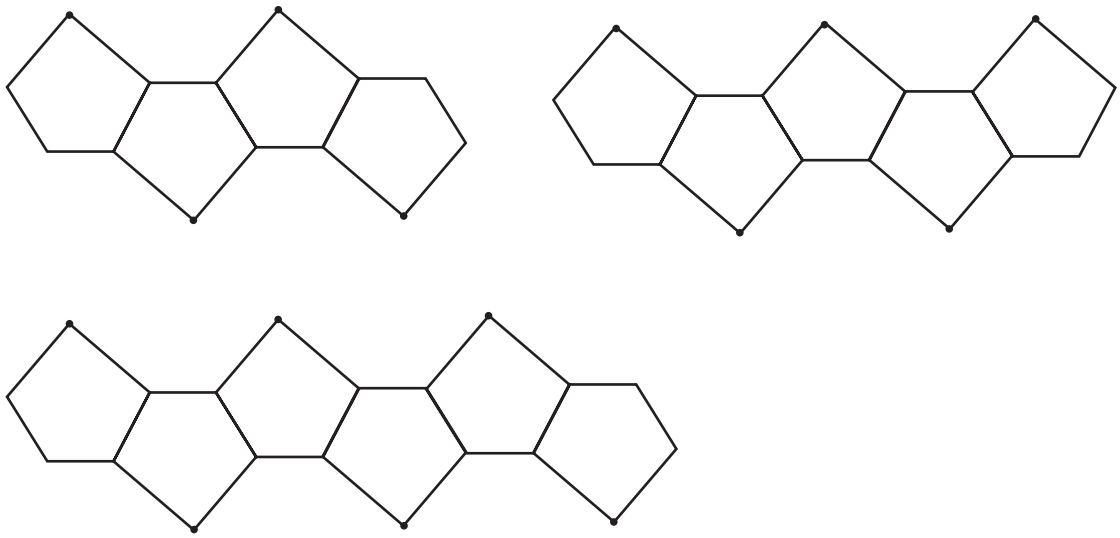
Q**6**. $T_n = (-2)^{n+1}$

$\Rightarrow T_1 = (-2)^{1+1} \quad = 4$

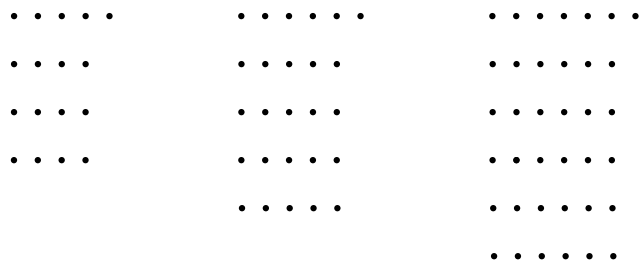
$T_6 = (-2)^{6+1} \quad = -128$

$T_{11} = (-2)^{11+1} = 4096.$

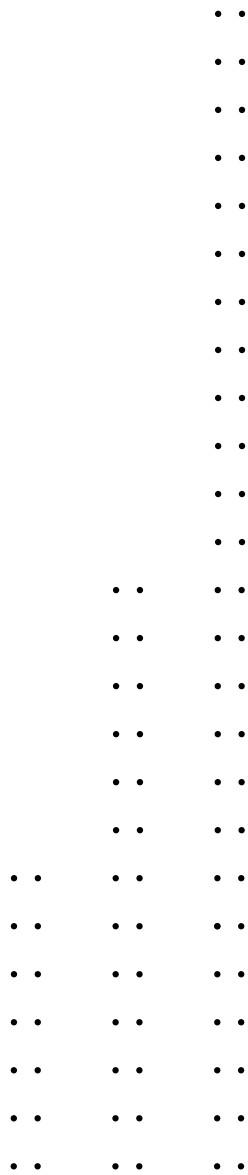
Q7. (i)



(ii)



(iii)



Q**8**. (i) $T_n = 4n - 2 = 2, 6, 10, 14, \ldots$ C

(ii) $T_n = 2n^2 = 2, 8, 18, 32, \ldots$ B

(iii) $T_n = n(n+1) = 2, 6, 12, 20, \ldots.$ D

(iv) $T_n = 2^n = 2, 4, 8, 16, \ldots.$ A

Q**9**. (i) $5, 6, 7, 8, 9$

$\Rightarrow T_1 = 5$

$T_2 = 6 = 5 + 1$

$T_3 = 7 = 5 + 2$

$T_4 = 8 = 5 + 3$

$T_5 = 9 = 5 + 4$

$\Rightarrow T_n \quad = 5 + (n-1) = 5 + n - 1 = n + 4.$

(ii) $2, 4, 6, 8, 10$

$\Rightarrow T_1 = 2$

$T_2 = 4 \quad = 2 \times 2$

$T_3 = 6 \quad = 2 \times 3$

$T_4 = 8 \quad = 2 \times 4$

$T_5 = 10 = 2 \times 5$

$\Rightarrow T_n \quad = 2 \times n = 2n.$

(iii) $2, 5, 8, 11, 14$

$\Rightarrow T_1 = 2$

$T_2 = 5 \quad = 2 + 3$

$T_3 = 8 \quad = 2 + 2(3)$

$T_4 = 11 = 2 + 3(3)$

$T_5 = 14 = 2 + 4(3)$

$\Rightarrow T_n \quad = 2 + (n-1)3 = 2 + 3n - 3 = 3n - 1.$

(iv) $1, 4, 9, 16, 25$

$T_1 = 1 \quad = 1^2$

$T_2 = 4 \quad = 2^2$

$T_3 = 9 \quad = 3^2$

$T_4 = 16 = 4^2$

$T_5 = 25 = 5^2$

$\Rightarrow T_n \quad = n^2.$

(v) $2, 5, 10, 17, 26$

$\Rightarrow T_1 = 2 \quad = 1+1 \quad = 1^2 + 1$

$T_2 = 5 \quad = 4+1 \quad = 2^2 + 1$

$T_3 = 10 = 9+1 \quad = 3^2 + 1$

$T_4 = 17 = 16+1 = 4^2 + 1$

$T_5 = 26 = 25+1 = 5^2 + 1$

$\Rightarrow T_n \qquad\qquad = n^2 + 1.$

(vi) $-1, 1, -1, 1, -1$

$\Rightarrow T_1 = -1$

$T_2 = \quad 1 = (-1)^2$

$T_3 = -1 = (-1)^3$

$T_4 = \quad 1 = (-1)^4$

$T_5 = -1 = (-1)^5$

$\Rightarrow T_n \qquad = (-1)^n.$

(vii) $1, 5, 9, 13, 17$

$\Rightarrow T_1 = 1$

$T_2 = 5 = 1 + 4$

$T_3 = 9 = 1 + 2(4)$

$T_4 = 13 = 1 + 3(4)$

$T_5 = 17 = 1 + 4(4)$

$\Rightarrow T_n \qquad = 1 + (n-1)4 = 1 + 4n - 4 = 4n - 3.$

(viii) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}$

$\Rightarrow T_1 = 1$

$T_2 = \frac{1}{2}$

$T_3 = \frac{1}{3}$

$T_4 = \frac{1}{4}$

$T_5 = \frac{1}{5}$

$\Rightarrow T_n = \frac{1}{n}.$

(ix) $\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}$

$\Rightarrow T_1 = \frac{2}{3} = \frac{n+1}{n+2}$

$T_2 = \frac{3}{4}$

$T_3 = \frac{4}{5}$

$T_4 = \frac{5}{6}$

$\Rightarrow T_n = \frac{n+1}{n+2}.$

(x) $(2\times3), (3\times4), (4\times5), (5\times6), \ldots.$

$\Rightarrow T_1 = 2\times3 = (n+1)(n+2)$

$T_2 = 3\times4$

$T_3 = 4\times5$

$T_4 = 5\times6$

$\Rightarrow T_n = (n+1)(n+2).$

Q**10**. $0, 1, 1, 2, 3, 5, 8, 13, 21 \ldots.$

The Fibonacci series is formed by adding the two previous terms to give the next term.

... 34, 55, 89, 144 ...

Q**11**.

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

(i) 1, 2, 3, 4, 5, ... $T_n = n$

(ii) 1, 3, 6, 10, 15, 21, ... $T_n = \frac{1}{2}(n^2 + n)$

(iii) 1, 2, 4, 8, 16, 32 ..... $T_n = \frac{2^n}{2} = 2^{n-1}$

(iv) 3, 6, 10, 15, 21, 28, ...

$= \frac{1}{2}(6, 12, 20, 30 \ldots..)$ $T_n = \frac{1}{2}(n+1)(n+2)$

Exercise 4.2

Q**1**. (i) $8, 13, 18, 23, ...$

$\Rightarrow a = 8,\ d = 5$

$\Rightarrow \quad T_n = a + (n-1)d$

$= 8 + (n-1)5$

$= 8 + 5n - 5 = 5n + 3$

also, $T_{22} = 5(22) + 3 = 113$

(ii) $16, 36, 56, 76, ...$

$\Rightarrow a = 16,\ d = 20$

$\Rightarrow \quad T_n = a + (n-1)d$

$= 16 + (n-1)20$

$= 16 + 20n - 20 = 20n - 4$

also, $T_{22} = 20(22) - 4 = 436$

(iii) $10, 7, 4, 1, ...$

$\Rightarrow a = 10,\ d = -3$

$\Rightarrow \quad T_n = a + (n-1)d$

$= 10 + (n-1)(-3)$

$= 10 - 3n + 3 \ = -3n + 13$

also, $T_{22} = -3(22) + 13 = -53$

Q**2**.

$$T_n = 5n - 2$$

$$\Rightarrow \left.\begin{aligned} T_1 &= 5(1) - 2 \ = 3 \\ T_2 &= 5(2) - 2 = 8 \\ T_3 &= 5(3) - 2 \ = 13 \\ T_4 &= 5(4) - 2 = 18 \end{aligned}\right\} 3, 8, 13, 18 \ ...$$

Q**3**. (i) $-5, -1, 3, 7, ... 75$

$\Rightarrow a = -5,\ d = 4$

$\Rightarrow T_n = -5 + (n-1)4$

$= -5 + 4n - 4 = 4n - 9$

when $T_n = 75 \quad \Rightarrow 75 = 4n - 9$

$4n = 75 + 9 = 84$

$n = \dfrac{84}{4} = 21.$

(ii) $2, 5, 8, 11, \ldots\ldots 59$

$\Rightarrow a = 2,\ d = 3$

$\Rightarrow T_n = a + (n-1)d$

$= 2 + (n-1)3$

$= 2 + 3n - 3 = 3n - 1$

when $T_n = 59 \Rightarrow 59 = 3n - 1$

$3n = 59 + 1 = 60$

$n = \frac{60}{3} = 20.$

(iii) $-\frac{3}{2}, -1, -\frac{1}{2}, 0 \ldots\ldots 14$

$\Rightarrow a = -\frac{3}{2},\ d = \frac{1}{2}$

$\Rightarrow T_n = a + (n-1)d$

$= -\frac{3}{2} + (n-1)\frac{1}{2}$

$= -\frac{3}{2} + \frac{n}{2} - \frac{1}{2} = \frac{n}{2} - 2$

when $T_n = 14 \quad \Rightarrow 14 = \frac{n}{2} - 2$

$\frac{n}{2} = 14 + 2 = 16$

$\Rightarrow \quad n = 16 \times 2 = 32.$

Q**4**. (i) $T_1 = 4, \quad T_7 = 22$

$\Rightarrow$ since $T_n = a + (n-1)d$

$T_1 = a + (1-1)d = a$

also, $T_7 = a + (7-1)d = a + 6d$

$\therefore \quad a = 4$

and $\quad a + 6d = 22$

$\Rightarrow \quad 4 + 6d = 22$

$6d = 18$

$\therefore \quad d = 3$

(ii) $\Rightarrow T_n = 4 + (n-1)3$

$= 4 + 3n - 3 = 3n + 1.$

$$\therefore \quad \left.\begin{aligned} T_1 &= 3(1) + 1 = 4 \\ T_2 &= 3(2) + 1 = 7 \\ T_3 &= 3(3) + 1 = 10 \\ T_4 &= 3(4) + 1 = 13 \\ T_5 &= 3(5) + 1 = 16 \end{aligned}\right\} 4, 7, 10, 13, 16 \ldots$$

(iii) Also, $T_{20} = 3(20) + 1 = 61.$

Q**5**. (i) red tiles sequence: $1, 2, 3, 4, \ldots\ldots T_n = n.$

orange tiles sequence: $8, 10, 12, \ldots\ldots \Rightarrow a = 8,\ d = 2.$

$$\Rightarrow T_n = 8 + (n-1)2$$
$$= 8 + 2n - 2$$
$$= 2n + 6.$$

$\therefore$ for design 8, $T_8 = 8$ red tiles

$T_8 = (2(8) + 6)$ orange tiles

$= 22$ orange tiles.

(ii) Total number of tiles needed

$= 9, 12, 15, 18, \ldots\ldots\ldots T_n = 3(n+2),$

i.e. the number of tiles is always a multiple of 3.

Since 38 is not a multiple of 3, no design will need 38 tiles.

Q**6**.

$$T_n = a + (n-1)d$$
$$\Rightarrow T_2 = a + (2-1)d = a + d$$
$$T_7 = a + (7-1)d = a + 6d$$
$$T_{13} = a + (13-1)d = a + 12d$$

$T_{13} = 27 \Rightarrow a + 12d = 27$

$T_7 = 3T_2 \Rightarrow a + 6d = 3(a + d)$

$\Rightarrow \quad a + 6d = 3a + 3d$

$\Rightarrow -2a + 3d = 0.$

Since $a + 12d = 27$ ..... A

and $-2a + 3d = 0$ ..... B

$\Rightarrow$ 2A: $2a + 24d = 54$

B: $-2a + 3d = 0$

$\Rightarrow \quad 27d = 54$ (adding)

$d = 2$

also, $-2a + 3(2) = 0$

$-2a = -6$

$a = 3$

$\therefore$ 3, 5, 7, 9, 11, 13 are the first six terms.

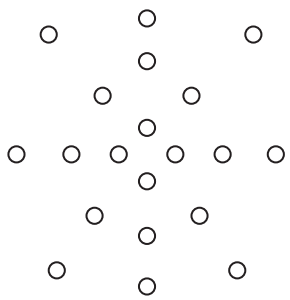
Q**7**. (i) $2k+2,\ 5k-3,\ 6k$ are three consecutive terms.

$$\begin{aligned}
\Rightarrow \quad 5k-3-(2k+2) &= 6k-(5k-3) \\
5k-3-2k-2 &= 6k-5k+3 \\
3k-5 &= k+3 \\
2k &= 8 \\
k &= 4.
\end{aligned}$$

(ii) $4p,\ -3-p,\ 5p+16$ are three consecutive terms.

$$\begin{aligned}
\Rightarrow \ -3-p-4p &= 5p+16-(-3-p) \\
-3-5p &= 5p+16+3+p \\
-3-5p &= 6p+19 \\
-11p &= 22 \\
p &= -2.
\end{aligned}$$

Q**8**.



(i) $12, 20, 28$

$\Rightarrow a=12,\ d=8$

$$\begin{aligned}
\Rightarrow T_n &= a+(n-1)d \\
&= 12+(n-1)8 \\
&= 12+8n-8 \ = 8n+4
\end{aligned}$$

(ii) $T_{15} = 8(15)+4 = 124$

(iii) $T_n = 164 \ \Rightarrow \ 8n+4 = 164$

$$\begin{aligned}
8n &= 160 \\
n &= 20
\end{aligned}$$

Q**9**. $T_n = 4n-2.$

$$\begin{aligned}
\Rightarrow T_{n+1} &= 4(n+1)-2 \\
&= 4n+4-2 = 4n+2.
\end{aligned}$$

$$\begin{aligned}
\therefore T_{n+1}-T_n &= (4n+2)-(4n-2) \\
&= \cancel{4n}+2-\cancel{4n}+2 = 4, \text{ a constant.}
\end{aligned}$$

Since there is a constant difference between terms, the sequence is arithmetic.

Q**10**. $T_n = n(n+2)$

$\Rightarrow T_{n+1} = (n+1)\big((n+1)+2\big)$

$= (n+1)(n+3)$

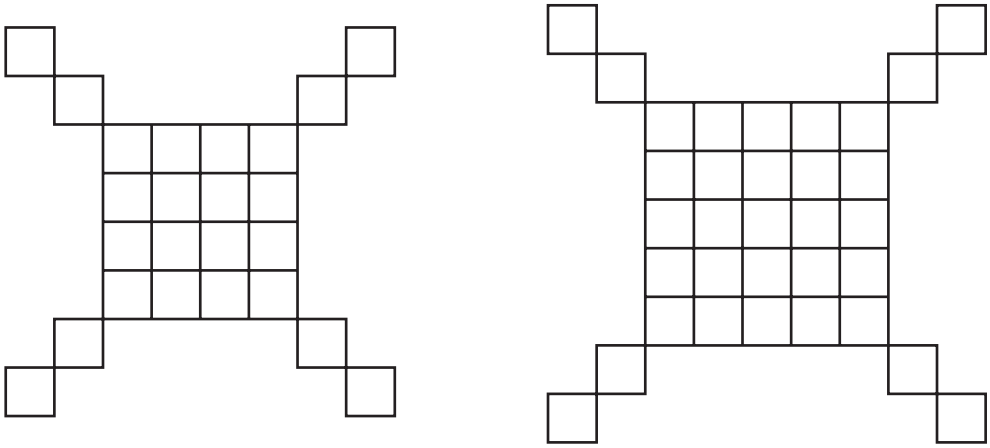
$\therefore T_{n+1} - T_n = (n+1)(n+3) - (n)(n+2)$

$= \cancel{n^2} + 3n + n + 3 - \cancel{n^2} - 2n$

$= 2n + 3$ which depends on n.

$\therefore$ Since the difference is <u>not</u> constant, the sequence is not arithmetic.

Q**11**.



(i) shape 7 will need 8 light-coloured tiles, since each design has 8 light-coloured tiles.

(ii) sequence for dark-coloured tiles is 1, 4, 9,.....$T_n = n^2$

$\Rightarrow$ for shape 7, $T_7 = 7^2$ dark tiles

$= 49$ dark tiles.

(iii) $T_n = n^2 + 8$

(iv) $T_{n+1} = (n+1)^2 + 8$

$= n^2 + 2n + 1 + 8$

$= n^2 + 2n + 9$

$\Rightarrow T_{n+1} - T_n = n^2 + 2n + 9 - (n^2 + 8)$

$= \cancel{n^2} + 2n + 9 - \cancel{n^2} - 8$

$= 2n + 1$ which depends on the value of n and is therefore not arithmetic.

Q12.

Number of hexagons	1	2	3	4	10	(20)	30
Perimeter	6	10	(14)	(18)	(42)	82	(122)

(i) 87 matchsticks left over

sequence: 6,10,14,18, ...

$\Rightarrow\ a = 6,\ d = 4$

$\Rightarrow T_n = 6 + (n-1)4$

$= 6 + 4n - 4$

$= 4n + 2.$

If $T_n = 4n + 2 = 87,$

$\Rightarrow 4n = 85$

$n = \dfrac{85}{4}$ which is not a whole number.

$\therefore$ Sandrine will have matchsticks left over.

(ii) $T_n = 4n + 2$

$T_{n+1} = 4(n+1) + 2$

$= 4n + 4 + 2 = 4n + 6$

$\Rightarrow T_{n+1} - T_n = 4n + 6 - (4n + 2)$

$= \cancel{4n} + 6 - \cancel{4n} - 2 = 4,$ a constant

$\therefore$ the sequence is arithmetic.

(iii) new design sequence: 6, 12, 18, ...

$\Rightarrow\ a = 6,\ d = 6.$

$\Rightarrow T_n = a + (n-1)d$

$= 6 + (n-1)6$

$= 6 + 6n - 6$

$= 6n$

Number of matches in each design: 6, 12, 18, 24, 30, 36, ...

Total number of matches used: 18, 36, 60, 90, 126, ...

$\therefore$ A total of 5 completed levels is possible using a total of 90 matchsticks with $122 - 90 = 32$ left over.

Q13.

$a = 12\,\text{min}$

$d = 6\,\text{min}$

$\Rightarrow T_n = 12 + (n-1)6$

$= 12 + 6n - 6$

$= 6n + 6.$

If $T_n = 60 \Rightarrow 6n + 6 = 60$

$\therefore\ 6n = 54$

$n = 9$ weeks.

Exercise 4.3

Q**1**. (i) $1+5+9+13$

$\Rightarrow\ a=1,\ d=4$

$$\therefore\ S_n=\frac{n}{2}(2a+(n-1)d)$$

$$=\frac{n}{2}(2(1)+(n-1)4)$$

$$=\frac{n}{2}(2+4n-4)$$

$$=\frac{n}{2}(4n-2)$$

$$\Rightarrow S_{20}=\frac{20}{2}(4(20)-2)$$

$$=780$$

(ii) $50+48+46+44$

$\Rightarrow\ a=50,\ d=-2$

$$\therefore\ S_n=\frac{n}{2}(2a+(n-1)d)$$

$$=\frac{n}{2}(2(50)+(n-1)(-2))$$

$$=\frac{n}{2}(100-2n+2)$$

$$=\frac{n}{2}(-2n+102)$$

$$\Rightarrow S_{20}=\frac{20}{2}(-2(20)+102)$$

$$=620$$

(iii) $1+1.1+1.2+1.3+\ .....$

$\Rightarrow\ a=1,\ d=0.1$

$$\Rightarrow\ S_n=\frac{n}{2}(2a+(n-1)d)$$

$$=\frac{n}{2}(2(1)+(n-1)(0.1))$$

$$=\frac{n}{2}(2+0.1n-0.1)$$

$$=\frac{n}{2}(0.1n+1.9)$$

$$\Rightarrow S_{20}=\frac{20}{2}((0.1)(20)+(1.9))$$

$$=39$$

(iv) $-7-3+1+5+ .....$

$\Rightarrow\ a=-7,\ d=4$

$$\Rightarrow\ S_n=\frac{n}{2}(2a+(n-1)d)$$
$$=\frac{n}{2}(2(-7)+(n-1)4)$$
$$=\frac{n}{2}(-14+4n-4)$$
$$=\frac{n}{2}(4n-18)$$
$$\Rightarrow S_{20}=\frac{20}{2}(4(20)-18)$$
$$=620$$

Q**2**. (i) $6+10+14+18+ ..... 50$

$\Rightarrow\ a=6,\ d=4$

$$\Rightarrow\ T_n=a+(n-1)d$$
$$=6+(n-1)4$$
$$=6+4n-4$$
$$=4n+2.$$

$T_n=50\ \Rightarrow 4n+2=50$

$$4n\quad=48$$
$$n\quad=12.$$

$$\therefore\ \ S_n=\frac{n}{2}(2a+(n-1)d)$$
$$\Rightarrow S_{12}=\frac{12}{2}(2(6)+(12-1)4)$$
$$=6(12+44)$$
$$=336$$

(ii) 1 + 2 + 3 + 4 + ... 100

It is obvious that $n=100$ (i.e. there are 100 terms in this sequence)

$a=1,\ d=1$

$$\therefore\ \ S_n=\frac{n}{2}(2a+(n-1)d)$$
$$\Rightarrow S_{100}=\frac{100}{2}(2(1)+(100-1)(1))$$
$$=50(2+99)$$
$$=5050.$$

(iii) $80+74+68+62+\ .....\ -34$

$\Rightarrow a=80,\ \ d=-6$

$\Rightarrow T_n=a+(n-1)d$

$=80+(n-1)(-6)$

$=80-6n+6$

$=-6n+86$

$T_n=-34\ \Rightarrow -6n+86=-34$

$-6n\quad=-120$

$n\quad=20$

$\therefore\ S_n=\frac{n}{2}(2a+(n-1)d)$

$\Rightarrow S_{20}=\frac{20}{2}(2(80)+(20-1)(-6))$

$=10(160-114)$

$=460$

Q**3**. $5+8+11+14+.....=98$

$\Rightarrow a=5,\ \ d=3$

$\therefore S_n=\frac{n}{2}(2a+(n-1)d)$

$\Rightarrow 98=\frac{n}{2}(2(5)+(n-1)3)$

$=\frac{n}{2}(10+3n-3)$

$=\frac{n}{2}(3n+7)$

$\therefore\ 196=3n^2+7n$

$\therefore\ 3n^2+7n-196=0$

To find ***n***;

$a=3,\ \ b=+7,\ \ c=-196$

$\therefore\ n=\frac{-7\pm\sqrt{(-7)^2-4(3)(-196)}}{2(3)}$

$=\frac{-7\pm\sqrt{2401}}{6}=\frac{-7\pm 49}{6}$

$=7\text{ or }\frac{-56}{6}\left(=-9\frac{2}{3}\right)$

Since n must be a natural number, $n=7$.

Q**4**. $T_n = 5 - 3n$

$\Rightarrow \quad T_1 = 5 - 3(1) \quad = 2 \quad = a$

also, $T_2 = 5 - 3(2) \quad = -1$

$\therefore \quad T_2 - T_1 = d \quad = -1 - 2 \quad = -3$

$$\Rightarrow \quad S_n = \frac{n}{2}(2a + (n-1)d)$$

$$\therefore \quad S_{10} = \frac{10}{2}(2(2) + (10-1)(-3))$$

$$= 5(4 - 27)$$

$$= -115$$

Q**5**. $a = 10,\ d = 2,\ S_n = 190$

$$\therefore \quad S_n = \frac{n}{2}(2a + (n-1)d)$$

$$\Rightarrow 190 = \frac{n}{2}(2(10) + (n-1)2)$$

$$\Rightarrow 380 = n(20 + 2n - 2)$$

$$380 = n(2n + 18)$$

$$= 2n^2 + 18n$$

$$\therefore\ 2n^2 + 18n - 380 = 0$$

$$\Rightarrow\ n^2 + \ 9n - 190 = 0$$

$$(n + 19)(n - 10) = 0$$

$$\therefore\ n = -19 \ \text{ or } \ n = 10$$

$$\therefore n = 10 \quad \text{since } n \in N.$$

Q**6**. (i) $$\sum_{r=1}^{6}(3r+1) = (3(1)+1) + (3(2)+1) + (3(3)+1) + (3(4)+1)$$
$$+ (3(5)+1) + (3(6)+1)$$
$$= 4 + 7 + 10 + 13 + 16 + 19$$
$$= 69$$

(ii) $$\sum_{r=0}^{5}(4r-1) = (4(0)-1) + (4(1)-1) + (4(2)-1) + (4(3)-1)$$
$$+ (4(4)-1) + (4(5)-1)$$
$$= -1 + 3 + 7 + 11 + 15 + 19$$
$$= 54$$

(iii) $\sum_{r=1}^{100} r \quad = \quad 1+2+3+4+ \ldots\ldots 100$

$\Rightarrow a=1,\ d=1,\ n=100$

$\therefore\ S_n = \frac{n}{2}(2a+(n-1)d)$

$= \frac{100}{2}(2(1)+(100-1)(1))$

$= 50(2+99)$

$= 50(101)$

$= 5050$

Q7. (i) $4+8+12+16+ \ldots. 124$

$\Rightarrow\ a=4,\ d=4$

$\therefore\ T_n = a+(n-1)d$

$\Rightarrow \quad = 4+(n-1)4$

$= \cancel{4}+4n-\cancel{4}$

$= 4n$

When $T_n = 124$: $\quad 4n = 124$

$n = 31$

$\therefore\ 4+8+12+16+ \ldots. 124 = \sum_{n=1}^{31} 4n$

(ii) $-10-9\frac{1}{2}-8-7\frac{1}{2}+ \ldots\ldots 4$

$\Rightarrow \quad a=-10,\ d=+\frac{1}{2}$

$\therefore\ T_n = a+(n-1)d$

$= -10+(n-1)\frac{1}{2}$

$= -10+\frac{n}{2}-\frac{1}{2}$

$= \frac{n}{2}-\frac{21}{2}$

When $T_n = 4$: $\frac{n}{2}-\frac{21}{2} = 4$

$\Rightarrow \quad n-21 = 8$

$\therefore \quad n \quad = 29$

$\therefore\ -10-9\frac{1}{2}-8-7\frac{1}{2}+ \ldots\ldots +4 = \sum_{n=1}^{29} \frac{1}{2}(n-21)$

(iii) $10+10.1+10.2+10.3+ \ldots\ldots 50$

$\Rightarrow\ a=10,\ d=0.1$

$\Rightarrow T_n = a+(n-1)d$

$T_n = 10+(n-1)(0.1)$

$= 10+0.1n-0.1$

$= 0.1n+9.9$

When $T_n = 50$: $\quad 0.1n+9.9=50$

$0.1n = 40.1$

$n = 401$

$\therefore\ \ 10+10.1+10.2+ \ldots\ldots 50 = \sum_{n=1}^{401} 0.1n+9.9$

Q**8**. $\quad T_4 = 15 \quad$ and also, $\quad S_5 = 55$

$T_n = a+(n-1)d \qquad S_n = \frac{n}{2}(2a+(n-1)d)$

$\Rightarrow T_4 = a+(4-1)d \qquad \Rightarrow S_5 = \frac{5}{2}(2a+(5-1)d)$

$\Rightarrow T_4 = a+3d \qquad \Rightarrow S_5 = \frac{5}{2}(2a+4d)$

$\therefore\ a+3d = 15$ .... A $\qquad S_5 = 5a+10d$

$\therefore\ 5a+10d = 55$ ..... B

$\therefore$ 5A: $5a+15d = 75$

B: $\underline{5a+10d = 55}$

$5d = 20$

$d = 4$

$\Rightarrow a+3(4) = 15$

$\therefore\ a = 3$

$\therefore$ The first five terms are 3, 7, 11, 15, 19.

Q**9**. Third term: $T_3 = 18$

Seventh term: $T_7 = 30$

$T_n = a + (n-1)d$

$\Rightarrow T_3 = a + (3-1)d$ also, $T_7 = a + (7-1)d$

$\Rightarrow T_3 = a + 2d$ $T_7 = a + 6d$

$T_3 = 18 \Rightarrow a + 2d = 18$

$T_7 = 30 \Rightarrow \underline{a + 6d = 30}$

$-4d = -12$ (subtracting)

$\therefore \quad d = 3$

$\Rightarrow a + 2(3) = 18$

$\therefore \quad a = 12$

$$\Rightarrow S_n = \frac{n}{2}(2a + (n-1)d)$$

$$\therefore S_{33} = \frac{33}{2}(2(12) + (33-1)3)$$

$$= \frac{33}{2}(24 + 96)$$

$$= 1980$$

Q**10**. (i) Number sequence for rings = 6,11,16, ...

$\Rightarrow a = 6, \ d = 5$

$T_n = a + (n-1)d$

$T_{10} = 6 + (10-1)5$

$= 51$ rings needed for design 10.

(ii) $T_{20} = 6 + (20-1)5$

$= 101$ rings needed for design 20.

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

$$\Rightarrow S_{20} = \frac{20}{2}(2(6) + (20-1)5)$$

$= 10(107)$

$= 1070$ rings needed for all 20 designs.

Q**11**. $-12, \ldots\ldots\ldots\ 40, \quad S_n = 196,$

$\Rightarrow a = -12.$

Also, $T_n = a + (n-1)d$

$\Rightarrow \quad 40 = -12 + (n-1)d$

$\therefore \quad 52 = (n-1)d$

$$S_n = \frac{n}{2}(2a + (n-1)d)$$

$$\Rightarrow 196 = \frac{n}{2}(2(-12) + (n-1)d)$$

$$= \frac{n}{2}(-24 + (n-1)d)$$

$$= \frac{n}{2}(-24 + 52)$$

$$\therefore\ 392 = 28n$$

$$\therefore\ n = 14.$$

Also, $52 = (14-1)d$

$52 = 13d$

$\therefore \quad d = 4.$

Q**12**. $1 + 2 + 3 + \ldots\ldots\ n$

$\Rightarrow a = 1,\ d = 1$

$$\therefore\ S_n = \frac{n}{2}(2a + (n-1)d)$$

$$= \frac{n}{2}(2(1) + (n-1)(1))$$

$$= \frac{n}{2}(2 + n - 1)$$

$$= \frac{n}{2}(n+1)$$

$\therefore\ 1 + 2 + 3 + \ldots\ldots\ 99 \qquad \Rightarrow n = 99.$

$$\therefore S_n = \frac{n}{2}(n+1)$$

$$\Rightarrow S_{99} = \frac{99}{2}(99+1)$$

$$= 4950$$

Q**13**. $T_{21} = 5\frac{1}{2}$

$S_{21} = 94\frac{1}{2}$

$T_n = a + (n-1)d \quad \Rightarrow T_{21} = a + (21-1)d = 5\frac{1}{2}$

$\therefore \quad a + 20d = 5\frac{1}{2}$

$\therefore \quad 2a + 40d = 11$ ..... A

Also, $S_n = \frac{n}{2}(2a + (n-1)d) \quad \Rightarrow S_{21} = \frac{21}{2}(2a + (21-1)d) = 94\frac{1}{2}$

$\therefore \quad \frac{21}{2}(2a + 20d) = 94\frac{1}{2}$

$\therefore \quad 42a + 420d = 189$

$\therefore \quad 14a + 140d = 63$ ..... B

7A: $14a + 280d = 77$

B: $\underline{14a + 140d = 63}$

$140d = 14$ (subtracting)

$\Rightarrow \quad d = 0.1$

$\therefore \; 2a + 40d = 11$

$\Rightarrow 2a + 40(0.1) = 11$

$\Rightarrow \quad a = 3.5$

$S_{30} = \frac{30}{2}(2(3.5) + (30-1)(0.1))$

$= 148.5$

Q**14**. $T_{21} = 37$

$S_{20} = 320$

$T_n = a + (n-1)d \Rightarrow T_{21} = a + (21-1)d = 37$

$\Rightarrow \quad a + 20d = 37$ ..... A

$S_n = \frac{n}{2}(2a + (n-1)d) \Rightarrow S_{20} = \frac{20}{2}(2a + (20-1)d) = 320$

$\Rightarrow \quad 10(2a + 19d) = 320$

$\Rightarrow \quad 2a + 19d = 32$ ..... B

2A: $2a + 40d = 74$

B: $\underline{2a + 19d = 32}$

$21d = 42$ (subtracting)

$\Rightarrow \quad d = 2$

$\therefore \quad a + 20d = 37$

$\Rightarrow \quad a + 20(2) = 37$

$a = 37 - 40 = -3$

$\therefore \quad S_{10} = \frac{10}{2}(2(-3) + (10-1)(2))$

$= 5(-6 + 18)$

$= 60$

Q**15**. $T_1, T_2, T_3, \ldots\ldots l$

$\therefore T_n = a + (n-1)d = l.$

$\Rightarrow \quad (n-1)d = l - a$

$S_n = \frac{n}{2}(2a + (n-1)d)$

$= \frac{n}{2}(2a + l - a)$

$= \frac{n(a+l)}{2} \qquad \left(i.e. \frac{n}{2}(a+l)\right)$

Q**16**. S_∞ of an arithmetic sequence cannot be found since

$S_n = \frac{n}{2}(a + l)$

$\Rightarrow S_\infty = \frac{\infty}{2}(a + l)$

But in an infinite sequence, there is no last term $\therefore$ the sum cannot be evaluated.

Exercise 4.4

Q**1**. (i) $3, 9, 27, 81, \ldots.$

$\frac{9}{3} = \frac{27}{9} = \frac{81}{27} = 3 = r \qquad \therefore$ sequence is geometric

$$\left.\begin{aligned} 81 \times 3 &= 243 \\ 243 \times 3 &= 729 \end{aligned}\right\} \; 3, 9, 27, 81, 243, 729$$

(ii) $1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \ldots$

$\frac{\frac{1}{3}}{1} = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{\frac{1}{27}}{\frac{1}{9}} = \frac{1}{3} = r \qquad \therefore$ sequence is geometric.

$$\left.\begin{aligned} \frac{1}{27} \times \frac{1}{3} &= \frac{1}{81} \\ \frac{1}{81} \times \frac{1}{3} &= \frac{1}{243} \end{aligned}\right\} \; 1, \frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \frac{1}{243}$$

(iii) $-1, 2, -4, 8, \ldots.$

$\frac{2}{-1} = \frac{-4}{2} = \frac{8}{-4} = -2 = r \quad \therefore$ sequence is geometric

$$\left.\begin{aligned} 8 \times -2 &= -16 \\ -16 \times -2 &= 32 \end{aligned}\right\} \; -1, 2, -4, 8, -16, 32$$

(iv) $1, -1, 1, -1, \ldots$

$\frac{-1}{1} = \frac{1}{-1} = \frac{-1}{1} = -1 = r \quad \therefore$ sequence is geometric

$$\left.\begin{aligned} -1 \times -1 &= 1 \\ 1 \times -1 &= -1 \end{aligned}\right\} \; 1, -1, 1, -1, 1, -1$$

(v) $1, 1\frac{1}{2}, 1\frac{1}{4}, 1\frac{1}{8}, \ldots$

$= 1, \frac{3}{2}, \frac{5}{4}, \frac{9}{8}$

$\Rightarrow \frac{\frac{3}{2}}{1} = \frac{3}{2}, \quad \frac{\frac{5}{4}}{\frac{3}{2}} = \frac{5}{6} \quad \therefore$ sequence is not geometric

(vi) $a, a^2, a^3, a^4, \ldots..$

$\Rightarrow \frac{a^2}{a} = \frac{a^3}{a^2} = \frac{a^4}{a^3} = a = r \quad \therefore$ sequence is geometric

$$\left.\begin{aligned} a^4 \times a &= a^5 \\ a^5 \times a &= a^6 \end{aligned}\right\} \; a, a^2, a^3, a^4, a^5, a^6$$

(vii) $1, 1.1, 1.21, 1.331, \ldots$

$\Rightarrow \frac{1.1}{1} = \frac{1.21}{1} = \frac{1.331}{1.21} = 1.1 = r \quad \therefore$ sequence is geometric

$$\left.\begin{array}{r} 1.331 \times 1.1 = 1.4641 \\ 1.4641 \times 1.1 = 1.61051 \end{array}\right\} 1, 1.1, 1.21, 1.331, 1.4641, 1.61051$$

(viii) $\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{36}, \ldots$

$\Rightarrow \frac{\frac{1}{6}}{\frac{1}{2}} = \frac{1}{3}, \ \frac{\frac{1}{12}}{\frac{1}{6}} = \frac{1}{2} \qquad \therefore$ sequence is not geometric

(ix) $2, 4, -8, -16, \ldots\ldots$

$\Rightarrow \frac{4}{2} \neq \frac{-8}{4} \neq \frac{-16}{-8} \qquad \therefore$ sequence is not geometric

(x) $\frac{3}{4}, \frac{9}{2}, 27, 162, \ldots$

$\Rightarrow \frac{\frac{9}{2}}{\frac{3}{4}} = \frac{27}{\frac{9}{2}} = \frac{162}{27} = 6 = r \qquad \therefore$ sequence is geometric

$$\left.\begin{array}{r} 162 \times 6 = \quad 972 \\ 972 \times 6 = 5,832 \end{array}\right\} \frac{3}{4}, \frac{9}{2}, 27, 162, 972, 5832$$

Q**2**. (i) $5, 10, \ldots\ldots$

$\Rightarrow \ a = 5, \ r = \frac{10}{5} = 2$

also, $T_n = a.r^{n-1}$

$\therefore T_{11} = 5.2^{(11-1)} = 5120$

(ii) $10, 25, \ldots.$

$\Rightarrow \ a = 10, \ r = \frac{25}{10} = 2.5$

also, $T_n = a.r^{n-1}$

$\therefore \quad T_7 = 10.(2.5)^{7-1}$

$= 10.(2.5)^6 = 2441.41$

(iii) $1.1, 1.21, \ldots\ldots$

$\Rightarrow \ a = 1.1, \ r = \frac{1.21}{1.1} = 1.1$

also, $T_n = a.r^{n-1}$

$\therefore \quad T_8 = 1.1(1.1)^{8-1}$

$= 1.1(1.1)^7 = 2.14358881$

(iv) $24, -12, 6, ...$

$\Rightarrow\ a = 24,\ r = \frac{-12}{24} = -\frac{1}{2}$

also, $T_n = a.r^{n-1}$

$\therefore \quad T_{10} = 24\left(-\frac{1}{2}\right)^{10-1}$

$= 24\left(-\frac{1}{2}\right)^{9} = \left(-\frac{3}{64}\right)$

Q**3**. $T_2 = 12,\ T_5 = 324$

$T_n = a.r^{n-1}$

$\Rightarrow T_2 = a.r^{2-1} = ar = 12$

$\Rightarrow T_5 = a.r^{5-1} = ar^4 = 324.$

$\therefore\ \frac{\cancel{a}\,r^{\cancel{4}\,3}}{\cancel{a}\,\cancel{r}} = \frac{324}{12}$

$\Rightarrow \quad r^3 = 27$

$\Rightarrow \quad r = 3$

$\therefore\ a(3) = 12$

$\Rightarrow a \quad = \frac{12}{3} = 4$

$\therefore$ sequence is $4,\ 4\times 3,\ 4\times 3^2,\ 4\times 3^3,\ 4\times 3^4$

$= 4,\ 12,\ 36,\ 108,\ 324$

Q**4**. $T_3 = 6,\ T_8 = 1458$

$T_n = a.r^{n-1}$

$\Rightarrow T_3 = a.r^{3-1} = ar^2 = 6$

$T_8 = a.r^{8-1} = ar^7 = 1458$

$\therefore\ \frac{\cancel{a}\,r^{\cancel{7}\,5}}{\cancel{a}\,\cancel{r^2}} = \frac{1458}{6}$

$\Rightarrow \quad r^5 = 243$

$\Rightarrow \quad r = \sqrt[5]{243} = 3$

Q**5**. $T_2 = 4,\ T_5 = -\dfrac{1}{16}$

$$T_n = a.r^{n-1}$$

$$\Rightarrow T_2 = a.r^{2-1} = ar = 4$$

$$T_5 = a.r^{5-1} = ar^4 = -\frac{1}{16}$$

$$\therefore\ \frac{\cancel{a}\,r^{\cancel{4}\,3}}{\cancel{a}\,\cancel{r}} = \frac{-\frac{1}{16}}{4} = -\frac{1}{64}$$

$$\therefore\quad r^3 = -\frac{1}{64}$$

$$\Rightarrow\quad r = \sqrt[3]{-\frac{1}{64}} = -\frac{1}{4}$$

$$\therefore\ a\left(-\frac{1}{4}\right) = 4$$

$$\Rightarrow a \quad = -16$$

$\therefore$ sequence is $-16, \left(-16\times -\dfrac{1}{4}\right), \left(-16\times\left(-\dfrac{1}{4}\right)^2\right), \left(-16\times\left(-\dfrac{1}{4}\right)^3\right), \left(-16\times\left(-\dfrac{1}{4}\right)^4\right)$

$$= -16, 4, -1, \frac{1}{4}, -\frac{1}{16}.$$

Q**6**. A: 2, 6, 8, ...

$\dfrac{6}{2} = \dfrac{18}{6} = 3 = r$ $\quad\therefore$ sequence is geometric

$T_4 = 18 \times 3 = 54$ dots

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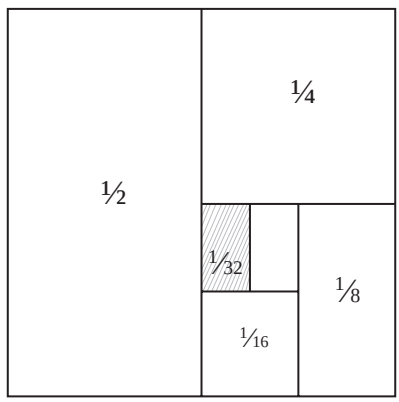
B: 3, 6, 10, ...

$\dfrac{6}{3} \neq \dfrac{10}{6}$ $\quad\therefore$ sequence is not geometric.

C: $1, \dfrac{1}{2}, \dfrac{1}{4}, \dfrac{1}{8}, \dfrac{1}{16}, ...$

$\dfrac{\frac{1}{2}}{1} = \dfrac{\frac{1}{4}}{\frac{1}{2}} = \dfrac{1}{2}$ $\quad\therefore$ sequence is geometric

$\therefore T_6 = \dfrac{1}{16}\times\dfrac{1}{2} = \dfrac{1}{32}$



D: $1, \frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \frac{8}{81} \ldots$

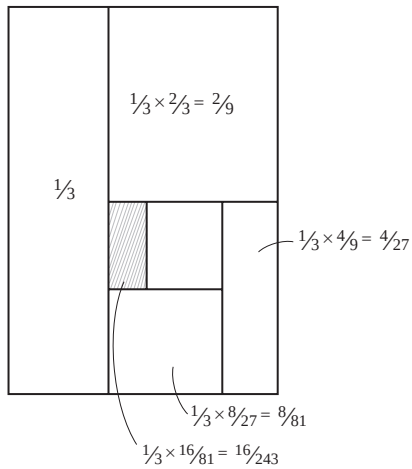
$\frac{\frac{1}{3}}{1} \neq \frac{\frac{2}{9}}{\frac{1}{3}}$ $\therefore$ sequence is not geometric

i.e $\frac{1}{3} \neq \frac{2}{3}$

[Note: if 1 is left out of this sequence,

$\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \frac{8}{81}$ is geometric;

$r = \frac{\frac{2}{9}}{\frac{1}{3}} = \frac{\frac{4}{27}}{\frac{2}{9}} = \frac{\frac{8}{81}}{\frac{4}{27}} = \frac{2}{3}$]



Q7. $n-2,\ n,\ n+3$ ..... geometric $\Rightarrow \frac{n}{n-2} = \frac{n+3}{n}$

$\therefore n^2 = (n+3)(n-2)$

$\cancel{n^2} = \cancel{n^2} + 3n - 2n - 6$

$\Rightarrow \quad n = 6$

$\Rightarrow \quad r = \frac{6}{6-2} = \frac{6}{4} = \frac{3}{2}$

$\therefore\ (6-2),\ 6,\ 6+3$

$\Rightarrow 4,\ 6,\ 9,\ \frac{27}{2}$

Q8. (i) $T_3 = -63,\ \ T_4 = 189$

$T_n = a.r^{n-1}$

$\Rightarrow T_3 = a.r^{3-1} = ar^2 = -63$

$T_4 = a.r^{4-1} = ar^3 = 189$

$\therefore\ \frac{\cancel{a}\, r^{\cancel{3}\,1}}{\cancel{a}\, \cancel{r^2}} = \frac{189}{-63}$

$\therefore \quad r = -3$

$\Rightarrow\ a(-3)^2 = -63$

$a = \frac{-63}{9} = -7$

(ii) $\therefore\ T_n = -7(-3)^{n-1}$

Q9. $T_1 = 16, \quad T_5 = 9$

$T_1 = a = 16$

$T_n = a.r^{n-1}$

$\Rightarrow T_5 = 16.r^{5-1} = 16.r^4 = 9$

$$r^4 = \frac{9}{16}$$

$$\Rightarrow \quad r = \sqrt[4]{\frac{9}{16}} = \frac{\sqrt{3}}{2}$$

$$\therefore \; T_7 = 16\left(\frac{\sqrt{3}}{2}\right)^{7-1} = \frac{16.\left(\sqrt{3}\right)^6}{2^6} = \frac{27}{4}$$

Q10. Let the first three terms be $\frac{a}{r}, a, ar$.

$$\Rightarrow \frac{a}{\cancel{r}}.a.a\cancel{r} = a^3 = 27$$

$$\Rightarrow a = \sqrt[3]{27} = 3$$

Also, $\frac{a}{r} + a + ar = 13$

$\Rightarrow \quad a + ar + ar^2 = 13r$

$a = 3: \quad 3 + 3r + 3r^2 = 13r$

$\Rightarrow \quad 3r^2 - 10r + 3 = 0$

$(3r-1)(r-3) = 0$

$\therefore \quad r = 1/3$ or $r = 3$

If $r = 3$ and $a = 3 \Rightarrow \quad \frac{3}{3}, 3, 3\times 3, 3\times 3^2$

$\Rightarrow \quad 1, 3, 9, 27$

If $r = \frac{1}{3}$ and $a = 3 \Rightarrow \quad \frac{3}{\frac{1}{3}}, 3, 3\times\left(\frac{1}{3}\right), 3\times\left(\frac{1}{3}\right)^2$

$\Rightarrow \quad 9, 3, 1, \frac{1}{3}$

Q11. $T_n = 3\times 2^{n-1}$

$$\Rightarrow \left.\begin{aligned} T_1 &= 3\times 2^{1-1} = 3\times 2^0 = 3 \\ T_2 &= 3\times 2^{2-1} = 3\times 2^1 = 6 \\ T_3 &= 3\times 2^{3-1} = 3\times 2^2 = 12 \\ T_4 &= 3\times 2^{4-1} = 3\times 2^3 = 24 \\ T_5 &= 3\times 2^{5-1} = 3\times 2^4 = 48 \end{aligned}\right\} \; 3, 6, 12, 24, 48$$

Q12. $T_n = 8\left(\frac{3}{4}\right)^n$

$$\Rightarrow T_1 = 8\left(\frac{3}{4}\right)^1 = 6$$

$$T_2 = 8\left(\frac{3}{4}\right)^2 = \frac{9}{2}$$

$$T_3 = 8\left(\frac{3}{4}\right)^3 = \frac{27}{8}$$

$$T_4 = 8\left(\frac{3}{4}\right)^4 = \frac{81}{32}$$

Q13. $T_n = (-1)^{n+1} \times \frac{5}{2^{n-4}}$

$$\Rightarrow \left.\begin{aligned} T_1 &= (-1)^{1+1} \times \frac{5}{2^{1-4}} = \frac{5}{2^{-3}} = 40 \\ T_2 &= (-1)^{2+1} \times \frac{5}{2^{2-4}} = \frac{-5}{2^{-2}} = -20 \\ T_3 &= (-1)^{3+1} \times \frac{5}{2^{3-4}} = \frac{5}{2^{-1}} = 10 \\ T_4 &= (-1)^{4+1} \times \frac{5}{2^{4-4}} = \frac{-5}{2^{0}} = -5 \end{aligned}\right\} 40, -20, 10, -5$$

Q**14**. (i) $x-3,\ x,\ 3x+4 \quad \Rightarrow \quad \frac{x}{x-3} = \frac{3x+4}{x}$

$$\therefore \quad x^2 = (3x+4)(x-3)$$

$$x^2 = 3x^2 - 9x + 4x - 12$$

$$\Rightarrow 2x^2 - 5x - 12 = 0$$

$$(2x+3)(x-4) = 0$$

$$\therefore x = -\frac{3}{2} \quad \text{or} \quad x = 4$$

If $x = 4$; $\quad 1,\ 4,\ 16,\ ....$

If $x = -\frac{3}{2}$; $\quad -\frac{9}{2}, -\frac{3}{2}, -\frac{1}{2},\ ...$

(ii) $x+1,\ x+4,\ 3x+2 \quad \Rightarrow \quad \dfrac{x+4}{x+1} = \dfrac{3x+2}{x+4}$

$$\therefore\ (x+4)(x+4) = (3x+2)(x+1)$$

$$\therefore\ x^2+8x+16 \quad = 3x^2+5x+2$$

$$\therefore\ 2x^2-3x-14 \ = 0$$

$$(2x-7)(x+2) = 0$$

$$\therefore\ x = 7/2 \text{ or } -2$$

If $x=-2$; $\quad -1,\ +2,\ -4, ...$

If $x = \dfrac{7}{2}$; $\quad \dfrac{9}{2}, \dfrac{15}{2}, \dfrac{25}{2}$

(iii) $x-2,\ x,\ x+3 \quad \Rightarrow \quad \dfrac{x}{x-2} = \dfrac{x+3}{x}$

$$\therefore \quad x^2 = (x+3)(x-2)$$

$$x^2 = x^2+x-6$$

$$\therefore \quad x-6 = 0$$

$$x \quad = 6.$$

When $x=6$; $\quad 4,\ 6,\ 9,\ .....$

(iv) $x-6, 2x, x^2 \quad \Rightarrow \quad \dfrac{2x}{x-6} = \dfrac{x^2}{2x}$

$$\Rightarrow \quad 4x^2 = x^2(x-6)$$

$$\Rightarrow \quad 4x^2 = x^3-6x^2$$

$$\therefore\ x^3-10x^2 \quad = 0$$

$$x^2(x-10) \quad = 0$$

$$\therefore \quad x=0 \text{ or } x=10$$

When $x=10$; $\quad 4,\ 20,\ 100, ...$

When $x=0$; $\quad$ no sequence is formed.

Q15. $\quad T_n = 2\times 3^n$

$$\Rightarrow T_{n+1} = 2\times 3^{n+1}$$

$$\therefore\ \frac{T_{n+1}}{T_n} = \frac{\cancel{2}\times 3^{n+1}}{\cancel{2}\times 3^n} = 3, \text{ a constant}$$

$\therefore$ Sequence is geometric.

Q16. $T_n = 3\times n^2$

$\Rightarrow T_{n+1} = 3\times(n+1)^2$

$\therefore \dfrac{T_{n+1}}{T_n} = \dfrac{3\times(n+1)^2}{3\times n^2} = \dfrac{3(n^2+2n+1)}{3n^2} = \dfrac{3n^2+6n+3}{3n^2}$

$= 1 + \dfrac{2}{n} + \dfrac{1}{n^2}$ which

is not constant $\therefore$ sequence is not geometric.

Q17. (i) $5, 15, 45, \ldots\ldots 3645$

$\Rightarrow a = 5,\ r = \dfrac{15}{5} = 3$

also, $T_n = a.r^{n-1}$

$= 5.3^{n-1}$

$\Rightarrow \quad 5.3^{n-1} = 3645$

$3^{n-1} = 729 = 3^6$

$\Rightarrow \quad n-1 = 6$

$n \quad = 7$

(ii) $48, 6, \dfrac{3}{4}, \ldots\ldots \dfrac{3}{2048}$

$\Rightarrow a = 48,\ r = \dfrac{6}{48} = \dfrac{1}{8}$

also, $T_n = a.r^{n-1}$

$= 48\left(\dfrac{1}{8}\right)^{n-1}$

$\Rightarrow 48\left(\dfrac{1}{8}\right)^{n-1} = \dfrac{3}{2048}$

$\left(\dfrac{1}{8}\right)^{n-1} = \dfrac{3}{48\times 2048} = \dfrac{1}{32768} = \dfrac{1}{8^5}$

$\therefore \quad n-1 = 5$

$n \quad = 6$

Q18. (i) $27\times\left(\dfrac{2}{3}\right)^1,\ 27\times\left(\dfrac{2}{3}\right)^2,\ 27\times\left(\dfrac{2}{3}\right)^3,\ 27\times\left(\dfrac{2}{3}\right)^4$

$= 18,\ 12,\ 8,\ \dfrac{16}{3}$

(ii) $T_n = 27\left(\dfrac{2}{3}\right)^n =$ height of the nth bounce

(iii) $T_{12} = 27\left(\dfrac{2}{3}\right)^{12} = 0.21\text{m}$

Q19. (i) $A = €4000(1.03)^t$

To find sum of money on deposit, let $t = 0$

$\Rightarrow \quad A = €4000(1.03)^0 = €4000$

(ii) year 1 : $A = €4000(1.03)^1 = €4120$

year 2 : $A = €4000(1.03)^2 = €4243.60$

year 3 : $A = €4000(1.03)^3 = €4370.91$

year 4 : $A = €4000(1.03)^4 = €4502.04$

(iii) year 10 : $A = €4000(1.03)^{10} = €5375.67$

(iv) double value = €8000

$\Rightarrow \quad €8000 = €4000(1.03)^t$

$\Rightarrow \quad 2 = (1.03)^t$

$t =$ between 23 and 24 years

$t = 23.45$

$$\left[\begin{array}{c}\text{Using logs: } \log 2 = \log 1.03^t = t\log 1.03 \\ t = \dfrac{\log 2}{\log 1.03} = 23.45\end{array}\right]$$

Q20. $A = P(1+i)^t$

$t = 10$ years, $P = €2500$, $A = €3047$

$\Rightarrow \quad 3,047 = 2,500\ (1+i)^{10}$

$1.2188 = (1+i)^{10}$

$(1+i) = \sqrt[10]{1.2188} = (1.2188)^{1/10} = 1.02$

$\Rightarrow \quad i = 0.02 \qquad = 2\%.$

Exercise 4.5

Q1. $2+6+18+54+....$ for 10 terms.

$\Rightarrow \ a = 2, \ r = \dfrac{6}{2} = 3$

also, $S_n = \dfrac{a(1-r^n)}{1-r}$

$$S_{10} = \frac{2(1-3^{10})}{1-3} = \frac{2(1-3^{10})}{-2} = 3^{10} - 1$$

$= 59,048$

Q2. $1024+512+256+ \ldots\ldots 32.$

$\Rightarrow\ a=1024,\ r=\dfrac{512}{1024}=\dfrac{1}{2}$

also, $T_n = a.r^{n-1}$

$$32=1024\left(\frac{1}{2}\right)^{n-1}$$

$$\Rightarrow\ \frac{32}{1024}=\left(\frac{1}{2}\right)^{n-1}$$

$$\Rightarrow\ \frac{1}{32}=\frac{1}{2^{n-1}}=\frac{1}{2^5}$$

$\Rightarrow\ n-1=5$

$n\ \ =6.$

[Note: $32=2^{n-1}$

$\Rightarrow \log 32=(n-1)\log 2$

$\Rightarrow \dfrac{\log 32}{\log 2}=n-1$

$5=n-1$]

Hence, $S_n=\dfrac{a(1-r^n)}{1-r}$

$$S_6=\frac{1024\left(1-\left(\frac{1}{2}\right)^6\right)}{1-\frac{1}{2}}$$

$$=2048\left(1-\left(\frac{1}{2}\right)^6\right)=2016$$

Q3. $1+2+4+8$

$\Rightarrow\ a=1,\ r=\dfrac{2}{1}=2$

$\therefore S_n=\dfrac{a(1-r^n)}{1-r}$

$$S_8=\frac{1(1-2^8)}{1-2}=2^8-1=255$$

Q4. $32+16+8+ \ldots\ldots$

$\Rightarrow\ a=32,\ r=\dfrac{16}{32}=\dfrac{1}{2}$

$\therefore S_n=\dfrac{a(1-r^n)}{1-r}$

$$\Rightarrow S_{10}=\frac{32\left(1-\left(\frac{1}{2}\right)^{10}\right)}{1-\frac{1}{2}}$$

$$=64\left(1-\left(\frac{1}{2}\right)^{10}\right)=63\frac{15}{16}=63.94$$

Q5. $4-12+36-108+ ....$

$\Rightarrow\ a=4,\ r=\dfrac{-12}{4}=-3$

$\therefore\ S_n=\dfrac{a(1-r^n)}{1-r}$

$\Rightarrow S_6=\dfrac{4(1-(-3)^6)}{1-(-3)}$

$=1-(-3)^6=-728$

Q6. $729-243+81-.....-\dfrac{1}{3}$

$\Rightarrow a=729,\ r=\dfrac{-243}{729}=\dfrac{-1}{3}$

also, $T_n=a.r^{n-1}$

$\therefore\ -\dfrac{1}{3}=729\left(-\dfrac{1}{3}\right)^{n-1}$

$\Rightarrow\ \dfrac{-1}{2187}=\dfrac{1}{(-3)^{n-1}}=\dfrac{-1}{3^7}=\dfrac{1}{(-3)^7}$ (or use logs)

$\Rightarrow\ n-1=7$

$n\ =8$

$\therefore\ S_n=\dfrac{a(1-r^n)}{1-r}$

$$S_8=\frac{729\left(1-\left(-\frac{1}{3}\right)^8\right)}{1-\left(-\frac{1}{3}\right)}$$

$$=546.75\left(1-\left(-\frac{1}{3}\right)^8\right)=546\frac{2}{3}$$

Q7. $\sum_{r=1}^{6}4^r=4^1+4^2+4^3+.....$

$=4+16+64$

$\Rightarrow a=4,\ r=\dfrac{16}{4}=4,\ n=6$

$\therefore\ S_n=\dfrac{a(1-r^n)}{1-r}$

$\Rightarrow S_6=\dfrac{4(1-4^6)}{1-4}$

$=\left(\dfrac{-4}{3}\right)\left(1-4^6\right)=5460$

Q8. $\sum_{r=1}^{8} 2\times 3^r = 2\times 3^1 + 2\times 3^2 + 2\times 3^3 + ...$

$$\Rightarrow a = 6,\ r = \frac{18}{6} = 3,\ n = 8$$

$$\therefore\ S_n = \frac{a(1-r^n)}{1-r}$$

$$\Rightarrow S_8 = \frac{6(1-3^8)}{1-3}$$

$$= -3(1-3^8) = 19680$$

Q9. $\sum_{r=1}^{10} 6\times\left(\frac{1}{2}\right)^r = 6\times\frac{1}{2} + 6\times\left(\frac{1}{2}\right)^2 + 6\times\left(\frac{1}{2}\right)^3 + ...$

$$= \quad 3 + \frac{6}{4} + \frac{6}{8} + ...$$

$$\Rightarrow \quad a = 3,\ r = \frac{1}{2},\ n = 10$$

$$\therefore\ S_n = \frac{a(1-r^n)}{1-r}$$

$$= \frac{3\left(1-\left(\frac{1}{2}\right)^{10}\right)}{1-\frac{1}{2}}$$

$$= 6\left(1-\left(\frac{1}{2}\right)^{10}\right)$$

$$= 5.994145$$

$$= 5.994$$

Q**10**. (i) $0.\dot{7} = 0.7777$

$$= \frac{7}{10} + \frac{7}{100} + \frac{7}{1000} + ...$$

$$\Rightarrow a = \frac{7}{10},\ r = \frac{\frac{7}{100}}{\frac{7}{10}} = \frac{1}{10}$$

$$\therefore\ S_\infty = \frac{a}{1-r}$$

$$= \frac{\frac{7}{10}}{1-\frac{1}{10}}$$

$$= \frac{7}{10}\times\frac{10}{9} = \frac{7}{9}$$

(ii) $0.\dot{3}\dot{5} = 0.353535$

$$= \frac{35}{100} + \frac{35}{10000} + \frac{35}{1000000} + ...$$

$$\Rightarrow a = \frac{35}{100},\ r = \frac{35}{10000} \times \frac{100}{35} = \frac{1}{100}$$

$$\therefore\ S_\infty = \frac{a}{1-r}$$

$$= \frac{\frac{35}{100}}{1-\frac{1}{100}}$$

$$= \frac{35}{100} \times \frac{100}{99} = \frac{35}{99}$$

(iii) $0.2\dot{3} = 0.23333$

$$= \frac{2}{10} + \frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + ...$$

$$= \frac{2}{10} + \left[\frac{3}{100} + \frac{3}{1000} + \frac{3}{10000} + ...\right.$$

Let $a = \frac{3}{100},\ r = \frac{3}{1000} \times \frac{100}{3} = \frac{1}{10}$

$$\therefore\ S_\infty = \frac{a}{1-r}$$

$$= \frac{\frac{3}{100}}{1-\frac{1}{10}}$$

$$= \frac{3}{100} \times \frac{10}{9} = \frac{1}{30}$$

$$\therefore 0.2\dot{3} = \frac{2}{10} + \left[\frac{1}{30}\right]$$

$$= \frac{7}{30}$$

(iv) $0.\dot{3}7\dot{0} = 0.370370370$

$$= \frac{370}{1000} + \frac{370}{1000000}$$

$$\Rightarrow\ a = \frac{370}{1000},\ r = \frac{\cancel{370}}{1000000} \times \frac{1000}{\cancel{370}} = \frac{1}{1000}$$

$$\therefore\ S_\infty = \frac{a}{1-r}$$

$$= \frac{\frac{370}{1000}}{1-\frac{1}{1000}}$$

$$= \frac{370}{1000} \times \frac{1000}{999} = \frac{370}{999} = \frac{10}{27}$$

(v) $0.1\dot{6}\dot{2} = 0.1626262$

$$= \frac{1}{10} + \left[\frac{62}{1000} + \frac{62}{100000} + \ldots\ldots\right]$$

Let $a = \frac{62}{1000},\ r = \frac{\cancel{62}}{100\cancel{0}\cancel{0}\cancel{0}} \cdot \frac{1\cancel{0}\cancel{0}\cancel{0}}{\cancel{62}} = \frac{1}{100}$

$$\therefore\ S_\infty = \frac{a}{1-r}$$

$$= \frac{\frac{62}{1000}}{1-\frac{1}{100}}$$

$$= \frac{62}{10\cancel{0}\cancel{0}} \cdot \frac{1\cancel{0}\cancel{0}}{99} = \frac{62}{990}$$

$$\therefore\ 0.1\dot{6}\dot{2} = \frac{1}{10} + \frac{62}{990}$$

$$= \frac{161}{990}$$

(vi) $0.3\dot{2}\dot{1} = 0.3212121$

$$= \frac{3}{10} + \left[\frac{21}{1000} + \frac{21}{100000} + \ldots\right]$$

Let $a = \frac{21}{1000}$, $r = \frac{\cancel{21}}{100\cancel{0}\cancel{0}\cancel{0}} \times \frac{1\cancel{0}\cancel{0}\cancel{0}}{\cancel{21}} = \frac{1}{100}$

$$\therefore \quad S_\infty = \frac{a}{1-r}$$

$$= \frac{\frac{21}{1000}}{1-\frac{1}{100}}$$

$$= \frac{\cancel{21}^{7}}{10\cancel{0}\cancel{0}} \times \frac{1\cancel{0}\cancel{0}}{\cancel{99}_{33}} = \frac{7}{330}$$

$$\therefore \quad 0.3\dot{2}\dot{1} = \frac{3}{10} + \frac{7}{330}$$

$$= \frac{106}{330} = \frac{53}{165}$$

Q11. $1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\ldots\ldots\left(\frac{1}{2}\right)^{n-1}$

$a=1,\ r=\frac{\frac{1}{2}}{1}=\frac{1}{2}$

$$\therefore\ S_n=\frac{a(1-r^n)}{1-r}$$

$$S_n=\frac{1\left(1-\left(\frac{1}{2}\right)^n\right)}{1-\frac{1}{2}}$$

$$=2\left(1-\left(\frac{1}{2}\right)^n\right)$$

$$=2-\frac{1}{2^{n-1}}$$

$$S_\infty=\lim_{n\to\infty}S_n=\lim_{n\to\infty}\left[2-\frac{1}{2^{n-1}}\right]$$

$$=2\quad\text{since }\lim_{n\to\infty}\frac{1}{2^{n-1}}=0.$$

$$S_\infty-S_n<0.001$$

$$\Rightarrow\quad 2-\left[2-\frac{1}{2^{n-1}}\right]<0.001$$

$$\Rightarrow\quad \frac{1}{2^{n-1}}<0.001$$

$$\Rightarrow\quad 2^{n-1}>\frac{1}{0.001}$$

$$2^{n-1}>1000$$

using logs: $\log 2^{n-1}>\log 1000$

$$(n-1)\log 2>\log 1000$$

$$n-1>\frac{\log 1000}{\log 2}$$

$$n-1>9.97$$

$$n>10.97$$

$$\therefore\quad n=11.$$

Exercise 4.6

Q**1**. (i) $5, 9, 13, 17, 21, ...$

1st difference = 4, 4, 4, 4, etc

$\therefore \quad T_n = 4n + a$

Since $T_1 = 4(1) + a = 5$

$\Rightarrow \quad a = 1$

$\therefore \quad T_n = 4n + 1$

(ii) $1, 4, 7, 10, 13, ...$

1st difference = $3, 3, 3, 3$, etc

$\therefore \quad T_n = 3n + a$

Since $T_1 = 3(1) + a = 1$

$\Rightarrow \quad a = -2$

$\therefore \quad T_n = 3n - 2$

(iii) $11, 16, 21, 26, 31, ...$

1st difference = 5, 5, 5, 5, etc

$\therefore \quad T_n = 5n + a$

Since $T_1 = 5(1) + a = 11$

$\Rightarrow \quad a = 6$

$\therefore \quad T_n = 5n + 6$

Q**2**. (i) $2, 1, 0, -1, -2, ...$

first difference = $-1 \quad \Rightarrow \; T_n = -n + a$

Since $T_1 = -(1) + a = 2$

$\Rightarrow \quad a = 3$

$\therefore \quad T_n = -n + 3$

(ii) $0, -2, -4, -6, -8, ...$

first difference = $-2 \quad \Rightarrow \; T_n = -2n + a$

Since $T_1 = -2(1) + a = 0$

$\Rightarrow \quad a = 2$

$\therefore \quad T_n = -2n + 2$

(iii) $-6, -4, -2, 0, 2, ...$

first difference =2 $\quad \Rightarrow \; T_n = 2n + a$

Since $T_1 = 2(1) + a = -6$

$\Rightarrow \quad a = -8$

$\therefore \quad T_n = 2n - 8$

Q**3**. (i) sequence of squares $= 1, 3, 6, 10$

first difference $= 2, 3, 4, ...$ etc

second difference $= 1, 1, 1, ...$ etc

$\therefore \quad T_n = an^2 + bn + c$, where $2a = 1$

$$\Rightarrow \quad a = \frac{1}{2}.$$

$$\therefore \quad T_n = \frac{1}{2}n^2 + bn + c$$

Since $T_1 = \frac{1}{2}(1)^2 + b(1) + c = 1$

$$\Rightarrow \quad b + c = \frac{1}{2}$$

$$\Rightarrow 2b + 2c = 1 \text{ ..... A}$$

also, $T_2 = \frac{1}{2}(2)^2 + b(2) + c = 3$

$$\Rightarrow \quad 2b + c = 1 \text{ .... B}$$

A : $\cancel{2b} + 2c = 1$

B : $\underline{\cancel{2b} + c = 1}$

A – B: $c = 0$ and $b + 0 = \frac{1}{2}$

$$\Rightarrow \quad b = \frac{1}{2}$$

$$\therefore \quad T_n = \frac{1}{2}n^2 + \frac{1}{2}n$$

$$\Rightarrow T_{28} = \frac{1}{2}(28)^2 + \frac{1}{2}(28)$$

$$= 406$$

$$\Rightarrow \text{area} = 406 \times 25\text{mm}^2$$

$$= 10{,}150\text{mm}^2$$

(ii) sequence for perimeter $= 4, 8, 12, 16$

first difference $= 4, 4, 4,$ etc

$\therefore \quad T_n = an + b$, where $a = 4$

$$= 4n + b$$

Since $T_1 = 4(1) + b = 4$

$$b = 0$$

$$\therefore \quad T_n = 4n$$

$$\Rightarrow T_{28} = 4 \times 28 = 112$$

$$\Rightarrow \text{Perimeter} = 112 \times 5\text{mm} = 560\text{mm}.$$

Q**4**. (i) $1, 4, 9, 16.....$

We recognise immediately that this is a sequence of squared numbers.

Alternatively, we could use differences.

$$\left.\begin{array}{ll} & 1,4,9,16 \\ 1^{st}\text{ difference:} & 3,5,7 \\ 2^{nd}\text{ difference:} & 2,2 \end{array}\right\} \begin{array}{l} \Rightarrow an^2+bn+c=T_n \\ \quad ,\text{where } 2a=2 \\ \quad \Rightarrow a=1 \\ \therefore\ n^2+bn+c \end{array}$$

We can then proceed that $b = c = 0$.

$\Rightarrow\ T_n = n^2$

$\therefore\ \ T_{30} = 30^2 = 900 \text{ Triangles} = 900 \text{ cm}^2$

(ii) $T_n = n^2 = 441$

$n = \sqrt{441} = 21$

Q**5**. (i) Areas $= (1\times 2), (2\times 3), (3\times 4)$

$= \mathrm{T}_1, \qquad \mathrm{T}_2, \qquad \mathrm{T}_3$

By inspection: $\mathrm{T}_n = n(n+1) = n^2+n$

$$\left.\begin{array}{ll} \textbf{or} & 2,6,12,20,.. \\ \text{first difference:} & 4,6,8 \\ \text{second difference:} & 2,2 \end{array}\right\} \begin{array}{l} \Rightarrow T_n = an^2+bn+c \\ \quad ,\text{where } 2a=2 \\ \quad \Rightarrow a=1 \\ \Rightarrow T_n = n^2+bn+c \end{array}$$

Since $T_1 = (1)^2 + b(1) + c = 2$

$\Rightarrow b + c = 1$ ..... A

$$\left.\begin{array}{l} \text{Also, } T_2 = (2)^2 + b(2) + c = 6 \\ \quad \Rightarrow 2b + c = 2 \text{ .... B} \\ \text{A:} \quad b + c = 1 \\ \text{B:} \quad \underline{2b + c = 2} \\ \text{A} - \text{B}: -b \quad = -1 \\ \quad b \quad = 1 \quad \Rightarrow c = 0 \end{array}\right\} \Rightarrow\ T_n = n^2 + n \ \ (\text{as before})$$

$\therefore\ T_{100} = 100^2 + 100$

$= 10{,}100 \text{ cm}^2$

(ii) $T_n = n^2 + n = 240$

$\Rightarrow\ n^2 + n - 240 = 0$

$(n+16)(n-15) = 0$

$\Rightarrow n = -16 \quad \text{or} \quad n = 15$

Since $n \in N$, $\quad n = 15$.

Q**6**. (i) Sequence: 6, 27, 74, 159, 294

1st difference: 21, 47, 85, 135

2nd difference: 26, 38, 50

3rd difference: 12, 12

$\Rightarrow T_n = an^3 + bn^2 + cn + d$

, where $6a = 12$

$\Rightarrow \quad a = 2$

$\therefore \quad T_n = 2n^3 + bn^2 + cn + d$

Since $T_1 = 2(1)^3 + b(1)^2 + c(1) + d = 6$

$b + c + d = 4$ ..... A

$T_2 = 2(2)^3 + b(2)^2 + c(2) + d = 27$

$4b + 2c + d = 11$ ..... B

$T_3 = 2(3)^3 + b(3)^2 + c(3) + d = 74$

$ab + 3c + d = 20$ C

A: $b + c + d = 4$

B: $\underline{4b + 2c + d = 11}$

A – B: $-3b - c = -7$ ..... D

B: $4b + 2c + d = 11$

C: $\underline{ab + 3c + d = 20}$

B – C: $-5b - c = -9$..... E

D: $-3b - c = -7$

E: $\underline{-5b - c = -9}$

D – E: $+2b = 2$

$\Rightarrow \quad b = 1$

D: $\therefore \; -3(1) - c = -7$

$\Rightarrow \quad c = 4$

A : $\therefore \; 1 + 4 + d = 4$

$\Rightarrow \quad d = -1$

$\Rightarrow T_n = 2n^3 + n^2 + 4n - 1$

[Note: We could have proceeded by subtracting the sequence $T_n = 2n^3$, i.e 2, 16, 54, 128, etc, and found quadratic element of pattern as before.]

(ii) Sequence: 3, -1, -1, 9, 35

1st difference: -4, 0, 10, 26

2nd difference: 4, 10, 16

3rd difference: 6, 6

$\Rightarrow T_n = an^3 + bn^2 + cn + d$

,where $6a = 6$

$\Rightarrow a = 1$

$\therefore\ T_n = n^3 + bn^2 + cn + d$

Sequence: 3, -1, -1, 9, 35

n^3 : 1, 8, 27, 64

Subtracting: 2, -9, -28, -55

$\Rightarrow\ bn^2 + cn + d$: 2, -9, -28, -55

1st difference: -11, -19, -27

2nd difference: -8, -8

$\Rightarrow 2b = -8$

$b = -4$

$\therefore\ T_n = n^3 - 4n^2 + cn + d$

Since $T_1 = (1)^3 - 4(1)^2 + c(1) + d = 3$

$\Rightarrow \qquad c + d = 6$ ..... A

$T_2 = (2)^3 - 4(2)^2 + c(2) + d = -1$

$2c + d = 7$ .... B

A: $c + d = 6$

B: $2c + d = 7$

A $-$ B: $-c = -1$

$\Rightarrow \quad c = 1 \ \therefore\ 1 + d = 6 \ \Rightarrow d = 5$

$\therefore\ T_n = n^3 - 4n^2 + n + 5$

[Note: Simultaneous equations in b,c,d could be used as in part (i) to evaluate coefficients of the quadratic part of T_n.]

(iii) Sequence: 3, -1, 2, 17, 50, 107 $\Rightarrow T_n = an^3 + bn^2 + cn + d$

1st difference: 3 15 33 57 , where $6a = 6$

2nd difference: 12 18 24 $\Rightarrow a = 1$

3rd difference: 6 6

$\therefore\ T_n = n^3 + bn^2 + cn + d$

Sequence: -1, 2, 17, 50, 107

n^3: 1, 8, 27, 64

$bn^2 + cn + d : -2,\ -6,\ -10,\ -14$ $\Rightarrow b = 0$

1st difference: $-4,\ -4,\ -4$ and $c = -4$

$\therefore\quad T_n = n^3 - 4n + d$

Since $T_1 = (1)^3 - 4(1) + d = -1$

$d = +2$

$\therefore\ T_n = n^3 - 4n + d$

[Note: Simultaneous equations in b, c, d could be used as in part (i) to evaluate coefficients of the quadratic part of T_n.]

Q7. (a) Sequence: 0, 3, 8, 15, ...

By inspection: $n^2 - 1$

$\Rightarrow\ T_n = n^2 - 1$

$\Rightarrow T_{24} = 24^2 - 1$

$= 575$ bright tiles

and 1 dark tile

Differences: 0, 3, 8, 15

1st difference: 3, 5, 7

2nd difference: 2, 2

$\Rightarrow T_n = an^2 + bn + c$

, where $2a = 2 \Rightarrow a = 1$

$\therefore\ T_n = n^2 + bn + c$

$T_1 = (1)^2 + b(1) + c = 0$

$\Rightarrow\ b + c = -1$ ..... A

$T_2 = (2)^2 + b(2) + c = 3$

$\Rightarrow 2b + c = -1$ ..... B

$\therefore$ B – A : $b = 0$

A: $b + c = -1$

$\Rightarrow c = -1$

$\therefore\ T_n = n^2 - 1$

(b) Sequence: 0, 2, 7, 14, ...

By inspection: $n^2 - 2$ $\left[\begin{array}{l}\text{We note that } n = 1 \text{ produces } 1^2 - 2 = -1 \\ \text{bright tiles! However, for } n = 2, 3, \ldots \\ \text{the formula fits exactly.}\end{array}\right]$

$\Rightarrow T_n = n^2 - 2$

$\Rightarrow T_{24} = 24^2 - 2$

$= 574$ bright tiles

and 2 dark tiles.

(c) Sequence: 0, 2, 6, 12,...

By inspection: $n^2 - n$

$\Rightarrow \quad T_n = n^2 - n$

$\Rightarrow \quad T_{24} = 24^2 - 24$

$= 552$ bright tiles

and 24 dark tiles.

Q**8**. (i) Sequence: 7, 16, 31, 52, 79 ..

1st difference: 9, 15, 21, 27 ...

2nd difference: 6, 6, 6 ....

$\Rightarrow T_n = an^2 + bn + c$

, where $2a = 6$

$\Rightarrow \; a = 3$

$\therefore \; T_n = 3n^2 + bn + c$

Since $T_1 = 3(1)^2 + b(1) + c = 7$

$\Rightarrow \; b + c = 4$ ..... A

Also, $T_2 = 3(2)^2 + b(2) + c = 16$

$2b + c = 14$ ..... B

$\Rightarrow$ B – A: $b = 0$

$\therefore$ A: $0 + c = 4$

$\Rightarrow \; c \quad = 4$

$\therefore \; T_n = 3n^2 + 4.$

(ii) Sequence: $1, 0, -3, -8, -15, ...$
1st difference: $-1, -3, -5, -7, ...$
2nd difference: $-2, -2, -2, ...$

$\Rightarrow T_n = an^2 + bn + c$
, where $2a = -2$
$\Rightarrow a = -1$

$\therefore T_n = -n^2 + bn + c$

Since $T_1 = -(1)^2 + b(1) + c \quad = 1$

$b + c \quad = 2$ ..... A

Also, $T_2 = -(2)^2 + b(2) + c \ = 0$

$2b + c \ = 4$ ..... B

$\Rightarrow$ B – A : $\quad b = 2$

$\therefore$ A : $\quad 2 + c = 2$

$\Rightarrow \quad c = 0$

$\therefore T_n = -n^2 + 2n.$

(iii) Sequence: $-1, 14, 53, 128, 251, ...$
1st difference: $15, 39, 75, 123, ...$
2nd difference: $24, 36, 48, ...$
3rd difference: $12, 12, ...$

$\Rightarrow T_n = an^3 + bn^2 + cn + d$
, where $6a = 12$
$\Rightarrow a = 2$

$\therefore T_n = 2n^3 + bn^2 + cn + d$

Sequence:	$-1, \ 14, \ 53, 128, \ 251$	
$2n^3$:	$2, \ 16, \ 54, 128, \ 250$	
$bn^2 + cn + d$:	$-3, -2, \ -1, \ 0, \ 1$	$\Rightarrow b = 0$
1st difference:	$1, \ 1, \ 1, \ 1$	$c = 1$

$\therefore \quad T_n = 2n^3 + n + d$

Since $T_1 = 2(1)^3 + (1) + d = -1$

$d = -4$

$\therefore \quad T_n = 2n^3 + n - 4.$

(iv) Sequence: $-2, 2, 6, 10, 14, ...$
1st difference: $4, 4, 4, 4, ...$

$\Rightarrow T_n = an + b$
, where $a = 4$

$\therefore T_n = 4n + b$

Since $T_1 = 4(1) + b = -2$

$b = -6$

$\therefore T_n = 4n - 6.$

(v) Sequence: 4, 31, 98, 223, 424, ...
1st difference: 27, 67, 125, 201, ...
2nd difference: 40, 58, 76, ...
3rd difference: 18, 18, ...

$T_n = an^3 + bn^2 + cn + d$
, where $6a = 18$
$\Rightarrow\ a = 3$

$\therefore\ T_n = 3n^3 + bn^2 + cn + d$

Sequence: 4, 31, 98, 223, 424
$3n^3$: 3, 24, 81, 192, 375

$bn^2 + cn + d$: 1, 7, 17, 31, 49
1st difference: 6, 10, 14, 18
2nd difference: 4, 4, 4

$\Rightarrow 2b = 4$
$b = 2.$

$\therefore\ T_n = 2n^2 + cn + d$

Since $T_1 = 2(1)^2 + c(1) + d = 1$
$c + d = -1$ ..... A

Also, $T_2 = 2(2)^2 + c(2) + d = 7$
$2c + d = -1$ .... B

$\therefore$ B – A: $c = 0$ $\therefore$ A: $0 + d = -1$
$\Rightarrow\ d = -1$

$\therefore\ T_n = 3n^3 + 2n^2 - 1.$

Revision Exercise (Core)

Q1. (i) $T_n = 3n + 4$

$\Rightarrow\ T_1 = 3(1) + 4 = 7$
$T_2 = 3(2) + 4 = 10$
$T_3 = 3(3) + 4 = 13$
$T_4 = 3(4) + 4 = 16$

7, 10, 13, 16

(ii) $T_n = 6n - 1$

$\Rightarrow\ T_1 = 6(1) - 1 = 5$
$T_2 = 6(2) - 1 = 11$
$T_3 = 6(3) - 1 = 17$
$T_4 = 6(4) - 1 = 23$

5, 11, 17, 23

(iii) $T_n = 2^{n-1}$

$$\Rightarrow \left.\begin{aligned} T_1 &= 2^{1-1} = 2^0 = 1 \\ T_2 &= 2^{2-1} = 2^1 = 2 \\ T_3 &= 2^{3-1} = 2^2 = 4 \\ T_4 &= 2^{4-1} = 2^3 = 8 \end{aligned}\right\} \quad 1, 2, 4, 8$$

(iv) $T_n = (n+3)(n+4)$

$$\Rightarrow \left.\begin{aligned} T_1 &= (1+3)(1+4) = 20 \\ T_2 &= (2+3)(2+4) = 30 \\ T_3 &= (3+3)(3+4) = 42 \\ T_4 &= (4+3)(4+4) = 56 \end{aligned}\right\} \quad 20, 30, 42, 56$$

(v) $T_n = n^3 + 1$

$$\Rightarrow \left.\begin{aligned} T_1 &= 1^3 + 1 = 2 \\ T_2 &= 2^3 + 1 = 9 \\ T_3 &= 3^3 + 1 = 28 \\ T_4 &= 4^3 + 1 = 65 \end{aligned}\right\} \quad 2, 9, 28, 65$$

Q2. $T_3 = 71, \quad T_7 = 55$

$T_n = a + (n-1)d, \quad a =$ first term, $d =$ common difference

$\Rightarrow T_3 = a + 2d = 71$ .... A

$\Rightarrow T_7 = a + 6d = 55$ .... B

$\Rightarrow$ A – B: $\quad -4d = 16$

$\Rightarrow \quad d = -4$

$\Rightarrow$ A: $\quad a + 2(-4) = 71$

$\Rightarrow a = 79$

$\Rightarrow$ first term = 79, common difference = –4.

Q3. $T_1 = 12, \quad S_\infty = 36$

$T_1 = a = 12$

$$S_\infty = \frac{a}{1-r} = 36$$

$$\Rightarrow \frac{12}{1-r} = 36$$

$$\Rightarrow 12 = 36 - 36r$$

$$r = \frac{24}{36} = \frac{2}{3}$$

Q4. (i) $-2,\ 4,\ -8,\ \ldots \quad \Rightarrow \quad r = \dfrac{4}{-2} = -2$

$$\therefore T_n = a.r^{n-1}$$
$$= -2(-2)^{n-1}$$

(ii) $1,\ \dfrac{1}{2},\ \dfrac{1}{4},\ \ldots. \qquad \Rightarrow \quad r = \dfrac{\frac{1}{2}}{1} = \dfrac{1}{2}$

$$\therefore T_n = a.r^{n-1}$$
$$= 1.\left(\frac{1}{2}\right)^{n-1} \quad \left[= \frac{1}{2^{n-1}} = 2^{1-n}\right]$$

(iii) $2,\ -6,\ 18, \ldots \quad \Rightarrow \quad r = \dfrac{-6}{2} = -3$

$$\therefore T_n = a.r^{n-1}$$
$$T_n = 2(-3)^{n-1}$$

Q5. (i) sequence : 12, 20, 28

1st difference: 8, 8 $\qquad \Rightarrow T_n = an + b$

$$T_n = 8n + b$$

$$\therefore T_n = 8n + b$$

Since $T_1 = 8(1) + b = 12,$

$$b = 4.$$

$$\therefore T_n = 8n + 4.$$

[Note: We observe an arithmetic sequence since there is a common difference $\Rightarrow T_n = a + (n-1)d$,where $a = 12$ and $d = 8$. $\quad \therefore T_n = 12 + (n-1)8$

$$= 8n + 4 \qquad]$$

(ii) $T_n = 8n + 4 = 2006$

$$8n = 2002$$

$$n = \frac{2002}{8} = 250 \cdot 25.$$

$\therefore$ The maximum number of complete cubes = 250.

Q6. (i) $T_2 = 21, \quad T_3 = -63$

$$r = \frac{T_{n+1}}{T_n} = \frac{T_3}{T_2} = \frac{-63}{21} = -3$$

$$T_n = a.r^{n-1}$$

$$T_2 = a.(-3)^{2-1} = 21$$

$$\Rightarrow \quad -3a = 21$$

$$a = -7.$$

(ii) $T_n = a.r^{n-1}$

$$T_1 = (-7).(-3)^{1-1}$$

$$= (-7).(-3)^0$$

$$= (-7).(1)$$

$$T_1 = -7$$

Q7. €2000 is invested at 2.5% compount interest.

$\Rightarrow$ After 1 year, amount on deposit, $A = 2000 + 2000(0.025)$

$$= 2000(1 + 0.025)^1$$

$$= 2000(1.025)^1$$

After 2 years, $A = 2000(1.025) + 2000(1.025)(0.025)$

$$= 2000(1.025)[1 + 0.025]$$

$$= 2000(1.025)^2$$

$\therefore$ After 5 years, $A = 2000(1.025)^5$.

Q8. 1 + 2 + 3 + 4 + ... 200.

$\Rightarrow \quad a = 1, \; d = 1, \; n = 200$

$$\therefore \quad S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

$$S_{200} = \frac{200}{2}\left(2(1) + (200-1)1\right)$$

$$= 100(201) \quad = 20{,}100$$

Q9. $T_5 = 2T_2$

$T_5 - T_2 = 9$

$$T_n = a + (n-1)d$$
$$\Rightarrow T_2 = a + (2-1)d = a + d$$
$$T_5 = a + (5-1)d = a + 4d$$

$$T_5 = 2T_2 \Rightarrow a + 4d = 2(a+d)$$
$$\Rightarrow a - 2d = 0 \text{ ..... A}$$
$$T_5 - T_2 = 9 \Rightarrow a + 4d - (a+d) = 9$$
$$\Rightarrow 3d = 9$$
$$d = 3$$
$$\Rightarrow \text{A: } a - 2(3) = 0$$
$$a = 6$$

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$
$$\Rightarrow S_{10} = \frac{10}{2}\left(2(6) + (10-1)3\right) = 195$$

Q10. $\sum_{r=3}^{16}(2r+1) = 2(3)+1, 2(4)+1, 2(5)+1 \text{ ..... } 2(16)+1$

$= 7,\ 9,\ 11,\ \text{.....}\ 33$

To find the number of terms n; $r = 3, 4, 5, \text{ ..... } 16$

$\Rightarrow n = 14 \ [= 16 - 3 + 1]$

or $T_n = a + (n-1)d$

$$\Rightarrow 7 + (n-1)2 = 33$$
$$\Rightarrow 2n + 5 = 33$$
$$n = 14$$

$$\therefore S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$
$$= \frac{14}{2}\left(2(7) + (14-1)2\right)$$
$$= 7(14 + 26) = 280$$

Revision Exercise (Advanced)

Q1. (i) $a = 2000$ lumens

$r = \frac{3}{5}$

$T_n = a.r^n$

$\Rightarrow T_{10} = 2000\left(\frac{3}{5}\right)^{10}$

$= 20$ lumens

(ii) $T_n = 2000(0.6)^n$

(iii) $\frac{1}{10}^{\text{th}}$ of original value = 200

$\therefore\ 2000(0.6)^n = 200$

$(0.6)^n = 0.1$

$\Rightarrow\ n\log(0.6) = \log(0.1)$

$n = \left(\frac{\log 0.1}{\log 0.6}\right) = 4.51$

$\Rightarrow$ After the 5th mirror.

Q2. $A = P(1+i)^t$

(i) original investment = €P

$\Rightarrow$ double value = €$2P$

$\therefore\ 2\cancel{P} = \cancel{P}(1+i)^t$

$\therefore\ 2 = (1+i)^t$

$\Rightarrow$ t for doubling depends only on i.

$\therefore\ \log 2 = \log(1+i)^t$

$\therefore\ \log 2 = t\log(1+i)$

$t = \frac{\log 2}{\log(1+i)}$

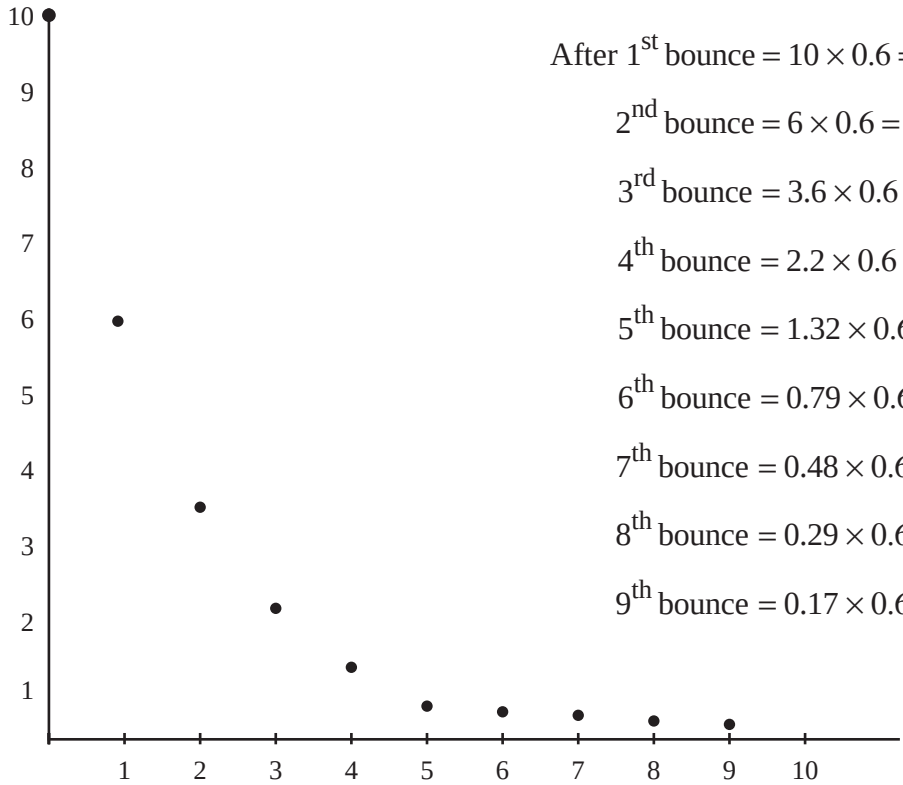
(ii) (a) $i = 2\%$ $\Rightarrow$ $t = \dfrac{\log 2}{\log(1+0.02)} = 35$ years

(b) $i = 5\%$ $\Rightarrow$ $t = \dfrac{\log 2}{\log(1+0.05)} = 14.2$ years

(c) $i = 10\%$ $\Rightarrow$ $t = \dfrac{\log 2}{\log(1+0.1)} = 7.3$ years

Q3. (i) 10 + 6 + 6 + 3.6 + 3.6 + 2.2 + 2.2 + ...

$10 \to 6 \Rightarrow r = \dfrac{6}{10} = 0.6$



After 1st bounce $= 10 \times 0.6 = 6$

2nd bounce $= 6 \times 0.6 = 3.6$

3rd bounce $= 3.6 \times 0.6 = 2.2$

4th bounce $= 2.2 \times 0.6 = 1.32$

5th bounce $= 1.32 \times 0.6 = 0.79$

6th bounce $= 0.79 \times 0.6 = 0.48$

7th bounce $= 0.48 \times 0.6 = 0.29$

8th bounce $= 0.29 \times 0.6 = 0.17$

9th bounce $= 0.17 \times 0.6 = 0.1$

(ii) $10 + 2(6 + 3.6 + 2.2 + \ldots.)$

An infinite geometric series.

(iii) Sequence: 6, 3.6, 2.2,...

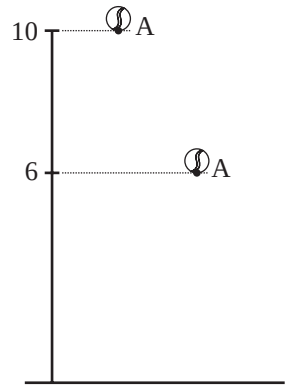
$\Rightarrow \quad a = 6, \qquad r = \dfrac{3.6}{6} = 0.6$

$S_\infty = \dfrac{a}{1-r} = \dfrac{6}{1-0.6} = 15$

$\Rightarrow$ distance travelled $= 10 + 2(15)$

$= 40$m.

(iv) Considering a point A on the bottom of the ball as the point to measure to, then the size of the ball has no effect on the answer.



Q4. (i) Sequence: 3, 6, 12, 24, 48

$$\Rightarrow\ a = 3,\ r = \frac{6}{3} = \frac{12}{6} = 2.$$

$$\therefore T_n = a.r^{n-1}$$
$$= 3.2^{n-1}$$

(ii) $T_n = 3.2^{n-1} > 1{,}000{,}000$

$$\Rightarrow 2^{n-1} > \frac{1{,}000{,}000}{3}$$

$$\therefore\ (n-1)\log 2 > \log 333{,}333.\dot{3}$$

$$n-1 > \frac{\log 333{,}333.\dot{3}}{\log 2}$$

$$n > 1 + 18.35$$

$$n > 19.35$$

$\Rightarrow$ the 20th term will exceed 1,000,000.

Q5. Sequence: 1, 2, 4, 8, 16 .....

$$\Rightarrow a = 1,\ r = \frac{2}{1} = \frac{4}{2} = 2.$$

$$\therefore\ T_n = a.r^{n-1}$$

(i) $T_{32} = 1.2^{32-1} = 2^{31} = 2147483648$ cent

$= €21{,}474{,}836$

(ii) $T_{64} = 1.2^{64-1} = 2^{63} = 9.22 \times 10^{18}$ cent

$= €9.22 \times 10^{16}$

Q6. Let sequence be $a-d,\ a,\ a+d$

$$\Rightarrow (a-d)+a+(a+d)=33$$

$$\Rightarrow \quad 3a = 33$$

$$a = 11$$

$$(a-d)(a)(a+d) = 935$$

$$\Rightarrow \quad (11-d)(11)(11+d) = 935$$

$$\Rightarrow \quad (11-d)(11+d) = 8$$

$$\Rightarrow \quad 121-d^2 = 8$$

$$d^2 = 36$$

$$d = \pm 6.$$

When $d=+6$, $\Rightarrow$ $(11-6),\ 11,\ (11+6)$ = $5, 11, 17$

When $d=-6$, $\Rightarrow$ $(11+6),\ 11,\ (11-6)$ = $17, 11, 5$

Q7. (i) Value of car $= €30,000$

After year 1 $= €30,000 - 30,000(0.13)$ $\quad [13\% = 0.13]$

$= 30,000(1-0.13)$

After year 2 $= 30,000(1-0.13) - 30,000(1-0.13)(0.13)$

$= 30,000(1-0.13)[1-0.13]$

$= 30,000(1-0.13)^2$

After year "a" $= 30,000(1-0.13)^a$

(ii) $a=5$: Value $= 30,000(1-0.13)^5$

$= €14,953$

(iii) $€30,000(1-0.13)^a < €6000$

$$(0.87)^a < \frac{\cancel{6000}^{\,1}}{\cancel{30,000}_{\,5}} = 0.2$$

$$a\log(0.87) < \log 0.2$$

$$a < \frac{\log 0.2}{\log 0.87}$$

$$a < 11.56$$

$\therefore$ during the 12th year.

Q8. $T_n = 3\left(\frac{2}{3}\right)^n - 1$

(i) $T_1 = 3\left(\frac{2}{3}\right)^1 - 1 = 1$

$T_2 = 3\left(\frac{2}{3}\right)^2 - 1 = \frac{1}{3}$

$T_3 = 3\left(\frac{2}{3}\right)^3 - 1 = -\frac{1}{9}$

(ii) $T_{n+1} = 3\left(\frac{2}{3}\right)^{n+1} - 1$

$= \cancel{3}\left(\frac{2}{3}\right)^n \cdot \frac{2}{\cancel{3}} - 1$

$= 2.\left(\frac{2}{3}\right)^n - 1$

(iii) $3T_{n+1} = 3\left(2\left(\frac{2}{3}\right)^n - 1\right)$

$2T_n = 2\left(3\left(\frac{2}{3}\right)^n - 1\right)$

$\therefore\ 3T_{n+1} - 2T_n = 3\left(2\left(\frac{2}{3}\right)^n - 1\right) - 2\left(3\left(\frac{2}{3}\right)^n - 1\right)$

$= \cancel{6\left(\frac{2}{3}\right)^n} - 3 - \cancel{6\left(\frac{2}{3}\right)^n} + 2$

$= -1$

$\Rightarrow \quad k = -1$

(iv) $\sum_{n=1}^{15}\left[3\left(\frac{2}{3}\right)^n-1\right]=\sum_{n=1}^{15}3\left(\frac{2}{3}\right)^n-15$

Sequence $=3\left(\frac{2}{3}\right)^1$, $3\left(\frac{2}{3}\right)^2$, $3\left(\frac{2}{3}\right)^3$, ...

$$\Rightarrow a=3\left(\frac{2}{3}\right),\quad r=\frac{3.\left(\frac{2}{3}\right)^2}{3.\left(\frac{2}{3}\right)}=\left(\frac{2}{3}\right)$$

$$=2$$

$$\therefore\ S_n=\frac{a\left(1-r^n\right)}{1-r}$$

$$=\frac{2\left(1-\left(\frac{2}{3}\right)^{15}\right)}{1-\frac{2}{3}}=6\left(1-\left(\frac{2}{3}\right)^{15}\right)$$

$$=5.98629$$

$$\therefore\sum_{n=1}^{15}\left[3\left(\frac{2}{3}\right)^n-1\right]=5.98629-15$$

$$=-9.014$$

Q9. (i)

$$S_n=T_1+T_2+T_3+\ ....\ T_{n-1}+T_n$$

$$\underline{S_{n-1}=T_1+T_2+T_3+\ ....\ T_{n-1}}$$

$$\therefore S_n-S_{n-1}=T_n$$

(ii) $S_n=3n^2+n$

$$\Rightarrow S_{n-1}=3(n-1)^2+(n-1)$$

$$\therefore T_n=S_n-S_{n-1}=3n^2+n-\left[3(n-1)^2+(n-1)\right]$$

$$=3n^2+n-\left[3\left(n^2-2n+1\right)+n-1\right]$$

$$=\cancel{3n^2}+\cancel{n}-\cancel{3n^2}+6n-3-\cancel{n}+1$$

$$=6n-2$$

(iii) $\sum_{r=1}^{n}(T_r)^2 = T_1^{\,2} + T_2^{\,2} + T_3^{\,2} + \ldots. T_n^{\,2}$

$$= (6-2)^2 + (6(2)-2)^2 + (6(3)-2)^2 + (6n-2)^2$$

$$= \sum (6n-2)^2$$

$$= \sum 36n^2 - 24n + 4$$

$$= 36\sum_{n=1}^{n} n^2 - 24\sum_{n=1}^{n} n + \sum_{n=1}^{n} 4$$

but $\sum \mathrm{n} = \frac{n}{2}(n+1)$, $\qquad \sum_{n=1}^{n} 4 = 4n.$

$$\sum \mathrm{n}^2 = \frac{n}{6}(2n+1)(n+1)$$

$$\sum (T_r)^2 = 36\left[\frac{n}{6}(2n+1)(n+1)\right] - 24\left[\frac{n}{2}(n+1)\right] + 4n$$

$$= n(n+1)[12n+6-12] + 4n$$

$$= n(n+1)(12n-6) + 4n$$

$$= n\left[(n+1)(12n-6) + 4\right]$$

$$= n\left[12n^2 + 6n - 2\right]$$

$$= 2n\left[6n^2 + 3n - 1\right]$$

Q10. $\log_4 x = \dfrac{\log_2 x}{\log_2 4}$

$$\begin{pmatrix} \text{let} \quad y = \log_2 4 \\ \Rightarrow 2^y = 4 = 2^2 \\ \Rightarrow \quad y = 2 \end{pmatrix}$$

$\therefore \log_4 x = \dfrac{\log_2 x}{\log_2 4} = \dfrac{\log_2 x}{2}$

Also, $\log_{16} x = \dfrac{\log_2 x}{\log_2 16}$

$$\begin{pmatrix} \text{let} \quad y = \log_2 16 \\ \Rightarrow 2^y = 16 = 2^4 \\ \Rightarrow \quad y = 4 \end{pmatrix}$$

$\therefore \log_{16} x = \dfrac{\log_2 x}{\log_2 16} = \dfrac{\log_2 x}{4}$

$\therefore \ \log_2 x, \log_4 x, \log_{10} x = \log_2 x, \dfrac{\log_2 x}{2}, \dfrac{\log_2 x}{4}$

$$\Rightarrow \frac{\frac{\log_2 x}{2}}{\log_2 x} = \frac{\frac{\log_2 x}{4}}{\frac{\log_2 x}{2}} = \frac{1}{2}$$

$\therefore \ r = \dfrac{1}{2}.$

$$S_\infty = \frac{a}{1-r} = \frac{\log_2 x}{1-\frac{1}{2}} = 2\log_2 x$$

$$= k\log_2 x$$

$$\Rightarrow k = 2.$$

Revision Exercise (Extended-Response Questions)

Q1. $T_n = an^3 + bn^2 + cn + d.$

(i) $\Rightarrow$ T_1, T_2, T_3, T_4, T_5

$a+b+c+d,\ 8a+4b+2c+d,\ 27a+9b+3c+d,\ 64a+16b+4c+d,\ 125a+25b+5c+d$

1st difference: $7a+3b+c,\ 19a+5b+c,\ 37a+7b+c,\ 61a+9b+c$

2nd difference: $12a+2b,\ 18a+2b,\ 24a+2b$

3rd difference: $6a,\ 6a$

(ii) The 3rd difference for all cubic sequences is always $6a$.

(iii) (a) The 2nd difference for all quadratic sequences is always $2a$.

(b) The 1st difference $(T_2 - T_1)$ for all quadratic sequences is always $3a+b$.

[Note: $T_n = an^2 + bn + c$

T_1 T_2 T_3 T_4

$a+b+c,\ 4a+2b+c,\ 9a+3b+c,\ 16a+4b+c$

1st difference: $3a+b,$ $5a+b,$ $7a+b$

2nd difference: $2a,$ $2a$]

(iv) T_1 T_2 T_3 T_4

5, 12, 25, 44

7, 13, 19 1st difference

6, 6 2nd difference

(v) $\therefore T_n = an^2 + bn + c$

,where $2a = 6$

$a = 3$

$\therefore T_n = 3n^2 + bn + c$

Also, $3a + b = 7$

$\therefore \quad 9 + b = 7 \quad \Rightarrow b = -2$

$\therefore \quad T_n = 3n^2 - 2n + c$

Since $T_1 = 3(1)^2 - 2(1) + c = 5$

$\Rightarrow \quad c = 4$

$\therefore \quad T_n = 3n^2 - 2n + 4.$

(vi) $\therefore \quad T_{20} = 3(20)^2 - 2(20) + 4.$

$= 1164.$

Q2. (i) Initial height = 40m;

after 1 bounce = $40.r^1$

after 2 bounces = $40.r^2$

after n bounces = $40.r^n$, where r is a fraction representing how far the ball bounces back up.

(ii) $T_n = 40r^n$

$$T_{10} = 40r^{10} = 1\text{m}$$

$$\Rightarrow \quad r^{10} = \frac{1}{40}$$

$$\Rightarrow \quad 10\log r = \log\frac{1}{40}$$

$$\log r = \frac{\log\frac{1}{40}}{10} = -0.1602$$

$$r = 10^{-0.1602} = 0.69 = 69\%$$

(iii) $T_1 = 40(0.69)^1 = 27.6\text{m}$

$T_2 = 40(0.69)^2 = 19.044\text{m}$

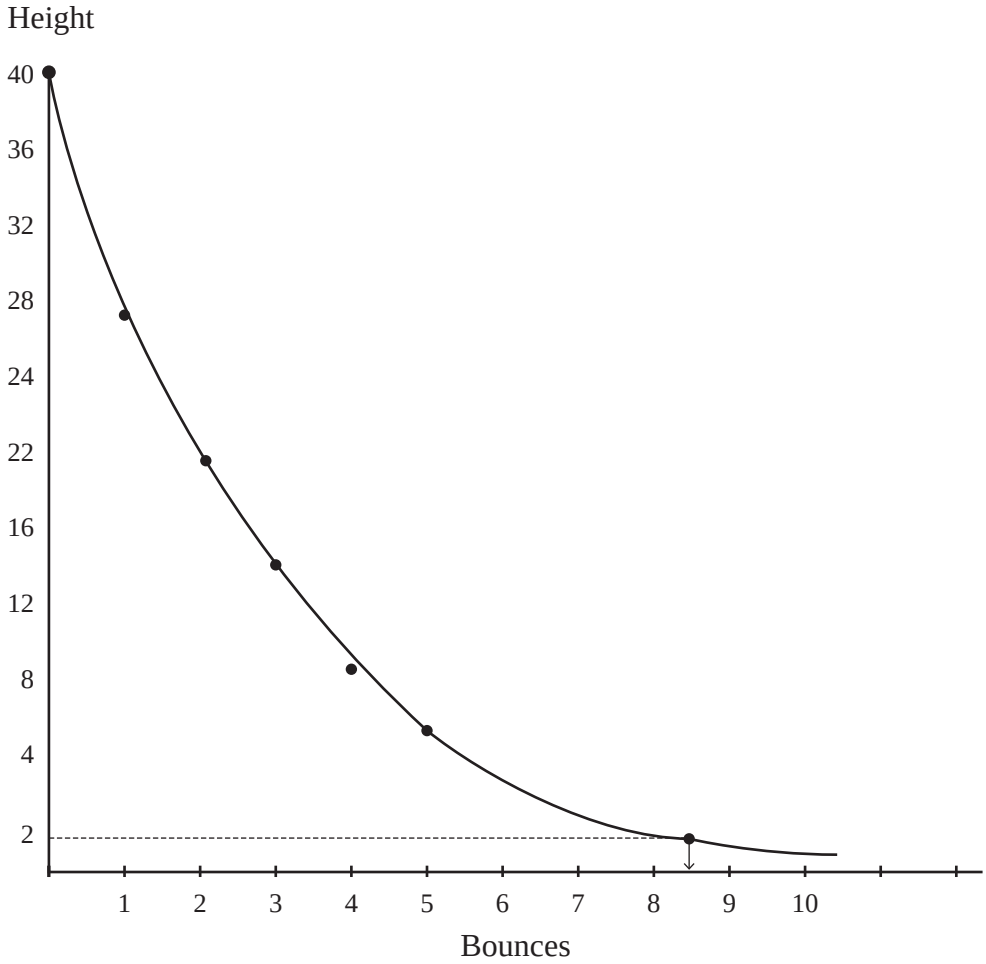
$T_3 = 40(0.69)^3 = 13.14\text{m}$

$T_4 = 40(0.69)^4 = 9.067\text{m}$

$T_5 = 40(0.69)^5 = 6.256\text{m}.$

Bounce	1st	2nd	3rd	4th	5th
Height	27.6	19.04	13.14	9.07	6.26

(iv)



(v) At least 9 bounces

(vi) $T_n = 40(0.69)^n < 2$

$$(0.69)^n < \frac{2}{40} = 0.05$$

$$n \log 0.69 < \log 0.05$$

$$n < \frac{\log 0.05}{\log 0.69} = 8.07$$

$\therefore$ 9 bounces are needed.

(vii) If conditions stayed exactly the same as on the first bounce, then the ball would continue to bounce for ever.
But in a real-life situation, the conditions do not same the same; the ball loses energy.
The ball may heat up and transfer energy to the ground. Also, sound energy may be dissipated.

Q3. Scheme1: €20, €22, €24, €26, ...

$$\Rightarrow a = €20, \quad d = 2.$$

$$T_n = a + (n-1)d$$

$$= 20 + (n-1)2$$

$$= 2n + 18 \quad = \quad \text{amount of money in week } n.$$

$$S_n = \frac{n}{2}\left(2a + (n-1)d\right)$$

$$= \frac{n}{2}\left(40 + (n-1)2\right)$$

$$= \frac{n}{2}(2n + 38)$$

$$= n(n+19) = \text{Total amount of money after week } n.$$

Scheme 2: €20, €20$\left(\frac{21}{20}\right)$, €20$\left(\frac{21}{20}\right)^2$, €20$\left(\frac{21}{20}\right)^3$, ...

$$\Rightarrow a = €20, \ r = \frac{21}{20}.$$

$$T_n = ar^{n-1}$$

$$= 20\left(\frac{21}{20}\right)^{n-1} = \text{amount of money in week } n.$$

$$S_n = \frac{a\left(1 - r^n\right)}{1 - r}$$

$$= 20\left(\frac{1 - \left(\frac{21}{20}\right)^n}{1 - \frac{21}{20}}\right)$$

$$= -400\left(1 - \left(\frac{21}{20}\right)^n\right)$$

$$S_n = 400\left(\left(\frac{21}{20}\right)^n - 1\right)$$

(ii) Scheme 1: $S_{36} = 36(36+19) = €1980$

Scheme 2: $S_{36} = 400\left(\left(\frac{21}{20}\right)^{36} - 1\right) = €1916.73$

$\Rightarrow$ Scheme 1 is better.

(iii) Assuming scheme 1: €$n(n+19) =$ total amount after n weeks

€7.50 spent per week $\Rightarrow$ €7.5n spent after n weeks.

$\therefore n(n+19) - 7.5n$ is saved after n weeks.

$\therefore n^2 + 19n - 7.5n = €400$

$\therefore n^2 + 11.5n - 400 = 0.$

$a = 1, \quad b = 11.5, \quad c = -400$

$$\Rightarrow n = \frac{-11.5 \pm \sqrt{(11.5)^2 - 4(1)(-400)}}{2}$$

$$= \frac{30.12}{2} \text{ or } \frac{-53.12}{2}$$

$$= 15.06 \text{ or } -26.5$$

$\therefore$ Ronan needs to save for 15 weeks and can buy the console in the 16th week.

Q4. (i) $160l$ at the start.

15% lost = 85% left at the end of the year.

$\Rightarrow \quad 160 \times 85\% = 136l$ left.

(ii) $2010 \rightarrow 2020 = 10$ years (end of year)

end of year 1 $= 160(0.85)^1$

end of year 2 $= 160(0.85)^2$

$\Rightarrow$ end of year 10 $= 160(0.85)^{10}$

$= 31.499$

$= 31.5l$

(iii) Last barrel left for 1 year $= 160(0.85) = 136\,l$ left

2nd last barrel left for 2 years $= 160(0.85)(0.85)$

$= 136(0.85)^2 = 115.6$ left

$\Rightarrow$ Total amount of liquid after 20 years

$= 160(0.85) + 160(0.85)^2 + 160(0.85)^3 + \ldots 160(0.85)^{20}$

$\Rightarrow \quad a = 160(0.85) = 136$

$r = 0.85$

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$= \frac{136\left(1-(0.85)^{20}\right)}{1-0.85}$$

$$= 871.52\,l$$

$$= 872\,l$$

Q5. (i) cost $= €15,000$

depreciation $= 20\%$ $\Rightarrow$ 80% of value left at the end of each year

$2005 \rightarrow 2007 = 2$years.

$\Rightarrow$ value $= €15,000(0.8)^2$

$= €9,600$

(ii) $€15,000(0.8) < €500$

$$\Rightarrow \quad (0.8)^n < \frac{\overset{1}{\cancel{5}}\,\cancel{0}\,\cancel{0}}{\underset{30}{\cancel{15}},\cancel{0}\,\cancel{0}\,\cancel{0}} = \frac{1}{30}$$

$$\Rightarrow \quad n\log(0.8) < \log\left(\frac{1}{30}\right)$$

$$n < \frac{\log\frac{1}{30}}{\log 0.8}$$

$$n < 15.242 \text{ years}$$

$\Rightarrow 2005 + 15 = 2020$

$\Rightarrow$ In year 2020.

(iii) Deposit $= €1000, \quad r = 5\% \ = 0.05$

$\Rightarrow$ savings $= 1000(1.05) + 1000(1.05)^2 + 1000(1.05)^3 + ...$

$...1000(1.05)^{15}$

$\Rightarrow a = 1000(1.05), \ r = 1.05$

$$\therefore \text{ Savings} = €1000(1.05)\frac{\left(1-(1.05)^{15}\right)}{(1-1.05)}$$

$$= €22,657.49$$

$$\simeq €22,657$$

(iv) Amount saved $= €22,657$

let inflation $= r\% = 0.0r$

$\Rightarrow$ cost of new machine after 15 years $= 15,000(1.05)^{15}$

$$\Rightarrow \quad 15,000(1.0r)^{15} = €22,657$$

$$(1.0r)^{15} = \frac{22,657}{15,000}$$

$$(1.0r)^{15} = 1.51$$

$$(1.0r) = (1.51)^{\frac{1}{15}}$$

$$1.0r = 1.028$$

$$\Rightarrow 0.0r = 0.028$$

$$r\% = 2.8\%$$