Christmas Test

2013

5th Year

Higher Level Maths

Q1. Solve $\left(1 - \frac{1}{x}\right)^2 = 2$

Give answers in form $a + \sqrt{b}$

$(a+b)^2 = a^2 + 2ab + b^2$

$\Rightarrow 1 - \frac{2}{x} + \frac{1}{x^2} = 2$

$\Rightarrow -1 - \frac{2}{x} + \frac{1}{x^2} = 0$

$\pm x^2 \pm 2x \mp 1 = 0$

$a = 1 \quad b = 2 \quad c = -1$

$$x = \frac{-2 \pm \sqrt{2^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$$

$\sqrt{\left(1 - \frac{1}{x}\right)^2} = \pm\sqrt{2}$

$1 - \frac{1}{x} = \pm\sqrt{2}$

$-\frac{1}{x} = -1 \pm \sqrt{2}$

$\frac{1}{x} = +1 \pm \sqrt{2}$

$x = \frac{1}{1 \pm \sqrt{2}}$

$x = \frac{1}{1+\sqrt{2}} = -1 + \sqrt{2}$

$x = \frac{1}{1-\sqrt{2}} = -1 - \sqrt{2}$

Q1 (b) Solve $x^3 + 3x^2 + x - 2 = 0$

Step 1 Guess

$f(-2) = (-2)^3 + 3(-2)^2 + (-2) - 2$

$= -8 + 12 - 2 - 2 = 0$

$\Rightarrow f(-2) = 0 \quad \Rightarrow x = -2$ soln

$\Rightarrow x + 2 =$ factor

Step 2 divide

$$\begin{array}{r l}
 & x^2 + x - 1 \\
x+2 \,\big) & \cancel{x^3} + 3x^2 + x - 2 \\
 & \mp\cancel{x^3} \mp 2x^2 \\
\hline
 & \cancel{x^2} + x \\
 & \mp\cancel{x^2} \mp 2x \\
\hline
 & \mp\cancel{x} \pm \cancel{2} \\
\hline
 & 0
\end{array}$$

Step 3 Solve quadratic

$x^2 + x - 1 = 0$

$a = 1 \quad b = 1 \quad c = -1$

$$x = \frac{-1 \pm \sqrt{1^2 - 4(1)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2}$$

Q2 (a)

$$2x - y + 3z = 20 \quad ①$$
$$7x + y + z = 23 \quad ②$$
$$3x + y - z = 3 \quad ③$$

① + ③ $\Rightarrow 5x + 2z = 23 \Rightarrow -10x - 4z = -46$

① + ② $\Rightarrow 9x + 4z = 43$

$$\begin{array}{r} -10x - 4z = -46 \\ 9x + 4z = 43 \\ \hline -x = -3 \\ x = 3 \end{array}$$

$5x + 2z = 23$

$\Rightarrow 5(3) + 2z = 23 \Rightarrow 2z = 8 \Rightarrow z = 4$

$\rightarrow$ ③ $3(3) + y - 4 = 3$

$y + 5 = 3 \Rightarrow y = -2$

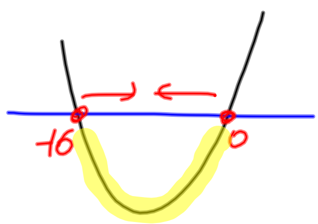
2(b) $(\underbrace{2-3k}_{a})x^2 + (\underbrace{4-k}_{b})x + \underbrace{2}_{c} = 0$ has no real roots.

If $\Delta = b^2 - 4ac < 0 \Rightarrow$ no real roots

$$\Delta = (4-k)^2 - 4(2-3k)(2) < 0$$
$$\cancel{16} - 8k + k^2 - \cancel{16} + 24k < 0$$
$$k^2 + 16k < 0$$
$$k(k+16) < 0$$

If $k(k+16) = 0$

$k = 0 \quad , \quad k = -16$



$-16 < k < 0$

Q3 (a) $z = -3 - i$, $i = \sqrt{-1}$

(i) $\bar{z} = -3 + i$

(ii) Quadratic equation with roots $\bar{z}$ and z.

$$x^2 - \begin{pmatrix} \text{sum} \\ \text{roots} \end{pmatrix} x + \begin{pmatrix} \text{product} \\ \text{roots} \end{pmatrix} = 0$$

Sum $= -3 - i - 3 + i = -6$

product $= (-3-i)(-3+i) = 9 \pm 1\cancel{i^2} = 10$

equation: $x^2 + 6x + 10 = 0$

3 (b) $az^3 + 22z^2 + bz + 40 = 0$. Find a & b and solve?

z is a root $\Rightarrow f(-3-i) = 0$

$\Rightarrow a(-3-i)^3 + 22(-3-i)^2 + b(-3-i) + 40 = 0$ * note: this is 2 equations because Re = Re & Im = Im

$(-3-i)^2 = 9 + 6i \mp 1\cancel{i^2} = 8 + 6i$

$(-3-i)^3 = (8+6i)(-3-i) = -24 - 8i - 18i \pm 6\cancel{i^2} = -18 - 26i$

$\Rightarrow a(-18-26i) + 22(8+6i) + b(-3-i) + 40 = 0$

$-18a - 26ai + 176 + 132i - 3b - bi + 40 = 0 + 0i$

Re = Re $\Rightarrow -18a + 176 - 3b + 40 = 0 \Rightarrow -18a - 3b + 216 = 0$

$\Rightarrow 18a + 3b = 216 \Rightarrow 6a + b = 72$ (1)

Im = Im $\Rightarrow -26ai + 132i - bi = 0i \Rightarrow 26a + b = 132$ (2)

(2) − (1) $\Rightarrow 20a = 60 \Rightarrow a = 3$

(1) $\Rightarrow 6(3) + b = 72 \Rightarrow b = 72 - 18 \Rightarrow b = 54$

$\Rightarrow 3z^3 + 22z^2 + 54z + 40 = 0$

Conjugate Root theorem $\Rightarrow$ 2 Solns are $z = -3 - i$ and $z = -3 + i$

the quadratic with these roots part a(ii) is $z^2 + 6z + 10$

divide

$$\begin{array}{r} 3z + 4 \\ z^2+6z+10 \overline{\big) \cancel{3z^3} + 22z^2 + 54z + 40} \\ \mp \cancel{3z^3} \mp 18z^2 \mp 30z \\ \hline \cancel{4z^2} + \cancel{24z} + \cancel{40} \\ \mp \cancel{4z^2} \mp \cancel{24z} \mp \cancel{40} \\ \hline \end{array}$$

$\Rightarrow$ 3rd soln $3z + 4 = 0 \Rightarrow z = -\frac{4}{3}$