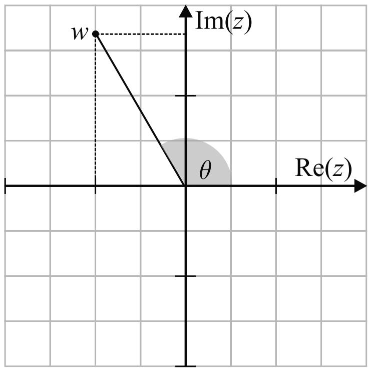
SEC Set D (SP 2014): Paper 1

Question 1 (25 marks)

Question 1 (a) (i)



$w=-1+\sqrt{3}i$

$r=|w|=\sqrt{(-1)^2+(\sqrt{3})^2}=\sqrt{1+3}=\sqrt{4}=2$

$|\tan\theta|=\left|\dfrac{\sqrt{3}}{-1}\right|=\sqrt{3}=\tan\alpha$, where α is the related angle in the first quadrant.

$\therefore \alpha=\tan^{-1}\sqrt{3}=60^\circ=\dfrac{\pi}{3}$

$\therefore \theta=\pi-\dfrac{\pi}{3}=\dfrac{2\pi}{3}$

$w=2\left[\cos\left(\dfrac{2\pi}{3}+2n\pi\right)+i\sin\left(\dfrac{2\pi}{3}+2n\pi\right)\right]$ in general polar form.

$\therefore w=2\left[\cos\left(\dfrac{2\pi+6n\pi}{3}\right)+i\sin\left(\dfrac{2\pi+6n\pi}{3}\right)\right]$

Question 1 (a) (ii)

$\boxed{[r(\cos\theta+i\sin\theta)]^n=r^n(\cos n\theta+i\sin n\theta)}$ (De Moivre's Theorem)

$z^2=-1+\sqrt{3}i\Rightarrow z=(-1+\sqrt{3}i)^{\frac{1}{2}}$

$z=2^{\frac{1}{2}}\left[\cos\left(\dfrac{2\pi+6n\pi}{3}\right)+i\sin\left(\dfrac{2\pi+6n\pi}{3}\right)\right]^{\frac{1}{2}}$

$=\sqrt{2}\left[\cos\left(\dfrac{2\pi+6n\pi}{6}\right)+i\sin\left(\dfrac{2\pi+6n\pi}{6}\right)\right]$ [There are 2 roots which you find by putting n = 0 and then n = 1.]

$n=0: z_1=\sqrt{2}\left[\cos\left(\dfrac{\pi}{3}\right)+i\sin\left(\dfrac{\pi}{3}\right)\right]$

$=\sqrt{2}(\frac{1}{2}+\frac{\sqrt{3}}{2}i)$

$n=1: z_2=\sqrt{2}\left[\cos\left(\dfrac{8\pi}{6}\right)+i\sin\left(\dfrac{8\pi}{6}\right)\right]$

$=\sqrt{2}\left[\cos\left(\dfrac{4\pi}{3}\right)+i\sin\left(\dfrac{4\pi}{3}\right)\right]$

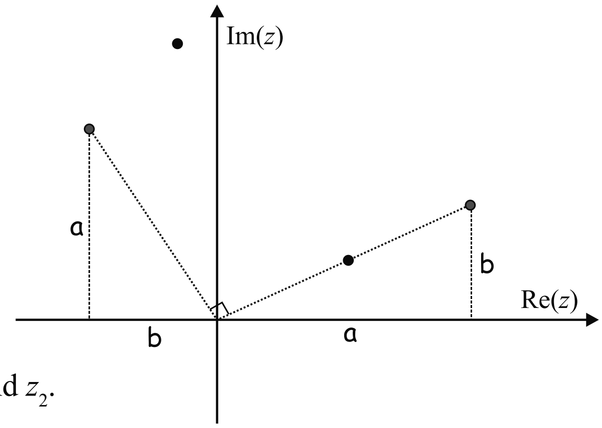
$=\sqrt{2}(-\frac{1}{2}-\frac{\sqrt{3}}{2}i)$

Question 1 (b)

(i) Let $z_1 = a + bi$.

$z_2 = iz_1 = i(a+bi) = ai + bi^2 = -b + ai$

Notice that the real and imaginary parts are swapped between z_1 and z_2. Can you pick out two points on the diagram where this might be the situation?



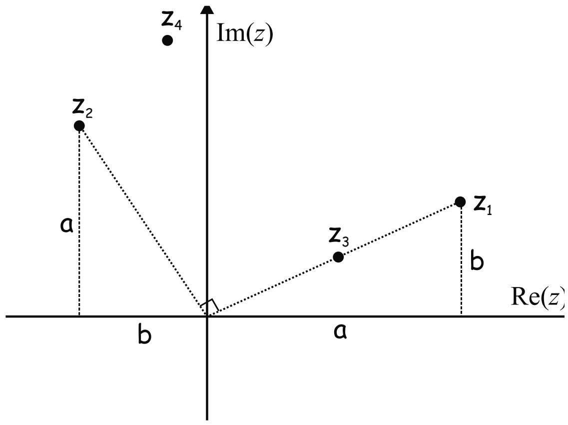
The two highlighted points seem to satisfy this condition.
At this stage I don't know which point is z_1 and z_2.

$z_3 = kz_1 = k(a+bi) = ka + kbi$

z_3 and z_1 are in a straight line with the origin.

$z_4 = z_2 + z_3 = (-b+ai) + (ka+kbi)$
$\quad = (ka-b) + (a+kb)i$

You are now in a position to mark in all the points.



(ii) $k \approx \frac{1}{2}$

Question 2 (25 marks)

Question 2 (a) (i)

Steps to Proof by Induction
1. Prove result is true for some starting value of $n \in \mathbb{N}$.
2. Assume result is true for $n = k$.
3. Prove result is true for $n = (k+1)$.

Required to Prove: $1+2+3+............+n = \dfrac{n(n+1)}{2}$

Step 1: Prove the result is true for $n = 1$.

$$1 = \frac{1(1+1)}{2}$$

$$1 = \frac{1(2)}{2}$$

$1 = 1$ [Therefore, true for $n = 1$.]

Step 2: Assume it is true for $n = k$.

$$\underline{1+2+3+............+k} = \frac{k(k+1)}{2}$$

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STEP 3: Prove it is true for $n = k + 1$.

Prove $\underline{(1+2+3+............+k)}+(k+1)=\dfrac{(k+1)(k+2)}{2}$

Use the result in Step 2 to prove Step 3.

$$(1+2+3+............+k)+(k+1)$$
$$=\frac{k(k+1)}{2}+(k+1)$$
$$=(k+1)\left[\frac{k}{2}+1\right]$$
$$=(k+1)\left[\frac{k+2}{2}\right]$$
$$=\frac{(k+1)(k+2)}{2}$$

Therefore, assuming true for $n = k$ means it is true for $n = k + 1$. So true for $n = 1$ and true for $n = k$ means it is true for $n = k + 1$. This implies it is true for all $n \in \mathbb{N}$.

Question 2 (a) (ii)

$$S_n=\frac{n(n+1)}{2}$$

$S_{100}-S_{50}=\dfrac{100(101)}{2}-\dfrac{50(51)}{2}=3775$ [This the the sum of the numbers between 51 and 100.]

QUESTION 2 (b)

$$\log_c \sqrt{x}+\log_c(cx)$$
$$=\log_c x^{\frac{1}{2}}+\log_c(cx)$$
$=\frac{1}{2}\log_c x+\log_c c+\log_c x$ [Using log rules 1 & 3.]
$$=\tfrac{3}{2}\log_c x+\log_c c$$
$$=\tfrac{3}{2}p+1$$

LOG RULES

$\log_a(xy)=\log_a x+\log_a y$.......**(1)**

$\log_a\left(\dfrac{x}{y}\right)=\log_a x-\log_a y$.......**(2)**

$\log_a(x^q)=q\log_a x$..................**(3)**

$\log_a 1=0$.................................**(4)**

$\log_a a=1$.................................**(5)**

$\log_b x=\dfrac{\log_a x}{\log_a b}$........................**(6)**

QUESTION 3 (25 MARKS)

Question 3 (a)

$$f(x) = x^3 + (1-k^2)x + k$$
$$f(-k) = (-k)^3 + (1-k^2)(-k) + k$$
$$= -k^3 - k + k^3 + k$$
$$= 0$$

> If $(x - k)$ is a factor of $f(x)$ then k is a root of $f(x) = 0$, i.e. $f(k) = 0$ and vice versa.

If $f(-k) = 0$, then $-k$ is a root of $f(x) = 0$.

Question 3 (b)

If $-k$ is a root, $(x + k)$ is a linear factor.

$x^3 + (1-k^2)x + k = (x+k)(x^2 + ax + 1)$ [A cubic is a linear multiplied by a quadratic.]

$$x^3 + 0x^2 + (1-k^2)x + k = x^3 + (a+k)x^2 + (ak+1)x + k$$
$$\therefore 0 = a + k \Rightarrow a = -k$$
$$\therefore x^3 + (1-k^2)x + k = (x+k)(x^2 - kx + 1)$$

$x^2 - kx + 1 = 0$ [Solve the quadratic equation using the formula.]

$$a = 1,\ b = -k,\ c = 1$$

$$x = \frac{-(-k) \pm \sqrt{(-k)^2 - 4(1)(1)}}{2(1)}$$
$$= \frac{k \pm \sqrt{k^2 - 4}}{2}$$

$$\boxed{x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}}$$

Question 3 (c)

$k^2 - 4 < 0$

$(k+2)(k-2) < 0$

[To have exactly one real root, the quadratic equation will give complex roots. This means the expression inside the square root must be negative.]

Solve the inequality above. First, solve the equality to locate the roots.

$$k^2 - 4 = 0$$
$$k^2 = 4$$
$$k = \pm 2$$

-3	$\boxed{-2}$	0	$\boxed{2}$	3
$(k+2)(k-2) < 0$		$(k+2)(k-2) < 0$		$(k+2)(k-2) < 0$
$(-1)(-5) < 0$		$(2)(-2) < 0$		$(5)(1) < 0$
False		True		False

ANSWER: $-2 < k < 2$

QUESTION 4 (25 MARKS)

Question 4 (a)

$2x+8y-3z=-1 ...(\mathbf{1})$ [Eliminate the x's by subtracting pairs of equations.]

$2x-3y+2z=2 .....(\mathbf{2})$

$2x+y+z=5 ....(\mathbf{3})$

$(\mathbf{1})-(\mathbf{2}): 11y-5z=-3 ...(\mathbf{4})(\times 4)$ [Eliminate the z's.]

$(\mathbf{1})-(\mathbf{3}): 7y-4z=-6 ....(\mathbf{5})(\times -5)$

$$\begin{array}{rl} 44y-20z & =-12 \\ -35y+20z & =30 \\ \hline 9y \qquad & =18 \Rightarrow y=2 \end{array}$$

$11(2)-5z=-3 ...(\mathbf{4})$ [Substitute the value of y into equation (**4**) to find z.]

$22-5z=-3$

$-5z=-25$

$\therefore z=5$

$2x+8(2)-3(5)=-1 ...(\mathbf{1})$ [Substitute the values of y and z into equation (**1**) to find x.]

$2x+16-15=-1$

$2x=-2$

$\therefore x=-1$

ANSWER: $(-1, 2, 5)$

QUESTION 4 (b) (i)

$f(x) = g(x)$ [Find the points where the graphs of the 2 functions intersect.]

$|x-3| = 2$

$x-3 = \pm 2$

$\therefore x = 1,\ 5$

Find where the $f(x)$ cuts the axes.

Cuts x-axis: Put $f(x) = 0$

$f(x) = 0$

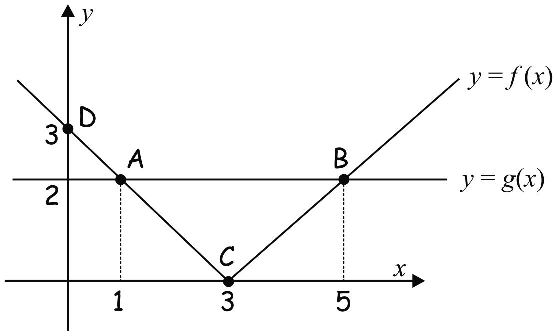
$|x-3| = 0$

$x-3 = 0$

$x = 3$

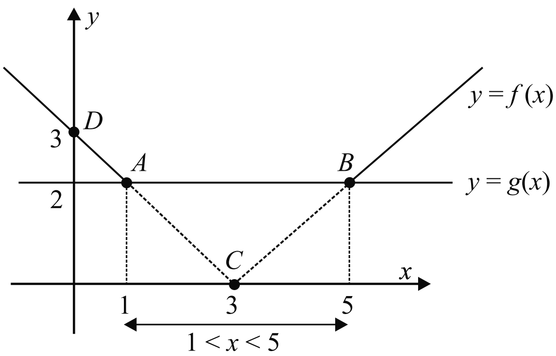
Cuts y-axis: Put $x = 0$

$f(0) = |0-3| = |-3| = 3$



ANSWERS: $A = (1, 2)$, $B = (5, 2)$, $C = (3, 0)$, $D = (0, 3)$

QUESTION 4 (b) (ii)



You can see from the graph the values of x where $f(x) < g(x)$.

ANSWER: $1 < x < 5$

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Question 5 (25 marks)

Question 5 (a)

Sketch the function by finding its turning points and end points.

$f(x) = x^3 - 5x^2 + 3x + 5$

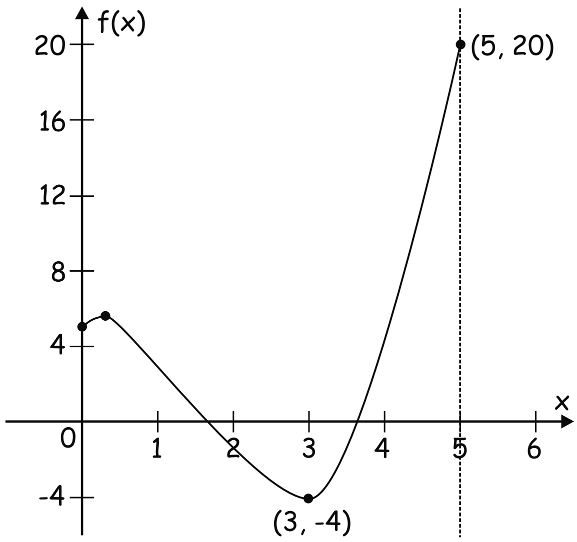
$f'(x) = 3x^2 - 10x + 3$

Turning points: $f'(x) = 0$

$3x^2 - 10x + 3 = 0$

$(3x - 1)(x - 3) = 0$

$x = \frac{1}{3}, 3$



$f(\frac{1}{3}) = (\frac{1}{3})^3 - 5(\frac{1}{3})^2 + 3(\frac{1}{3}) + 5 = 0 \Rightarrow (\frac{1}{3}, \frac{148}{27})$ is a turning point.

$f(3) = (3)^3 - 5(3)^2 + 3(3) + 5 = 0 \Rightarrow (3, -4)$ is a turning point.

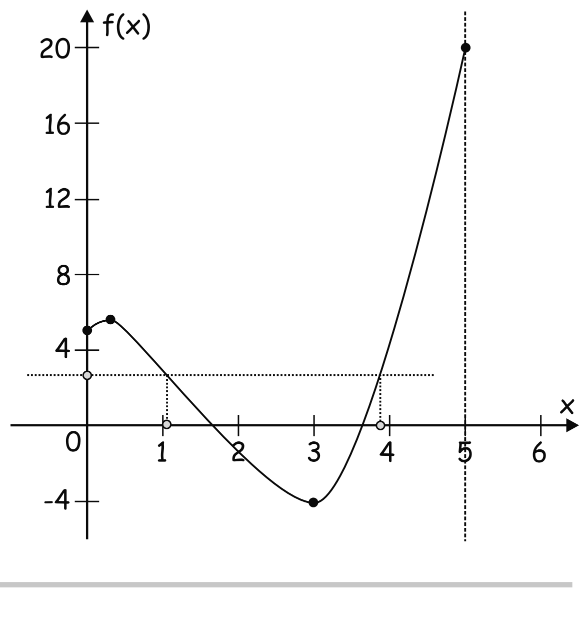
$x = 0 : f(0) = (0)^3 - 5(0)^2 + 3(0) + 5 = 5 \Rightarrow (0, 5)$ is the starting point.

$x = 5 : f(5) = (5)^3 - 5(5)^2 + 3(5) + 5 = 20 \Rightarrow (5, 20)$ is the finishing point.

Answer: Maximum value of $f = 20$
Minimum value of $f = -4$

Question 5 (b)

The function f is **not** injective.
An injective function never maps distinct elements of its domain to the same element of its range. Each element in the domain must map on to a unique element in the range.
Look at the sketch of a diagram of the function f.



You can see from the sketch that the function is not injective as two points are shown on the x-axis which map on to the same point on the $f(x)$ axis.

Question 6 (25 marks)

Question 6 (a)

(i) **1**. $I_1 = \frac{1}{4}x^4 + x^3 + 3x + 1$

2. $I_2 = \frac{1}{4}x^4 + x^3 + 3x + 2$

3. $I_3 = \frac{1}{4}x^4 + x^3 + 3x + 3$

$$\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + c}$$

(ii) $\int f(x)\,dx = h(x)$

$h(x)$ is an indefinite integral which means that when you differentiate it with respect to x you get $h(x)$

or

$h'(x) = f(x)$.

(iii) $I = \frac{1}{4}x^4 + x^3 + 3x + c$

Question 6 (b)

(i) Let $h(x) = x\ln x$, for $x \in \mathbb{R}$, $x > 0$.

Find $h'(x)$.

(ii) Hence, find $\int \ln x\,dx$.

Solution

Product Rule

$$\boxed{y = uv \Rightarrow \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}}$$

$$\boxed{y = \ln x \Rightarrow \frac{dy}{dx} = \frac{1}{x}}$$

(i) $h(x) = x\ln x$

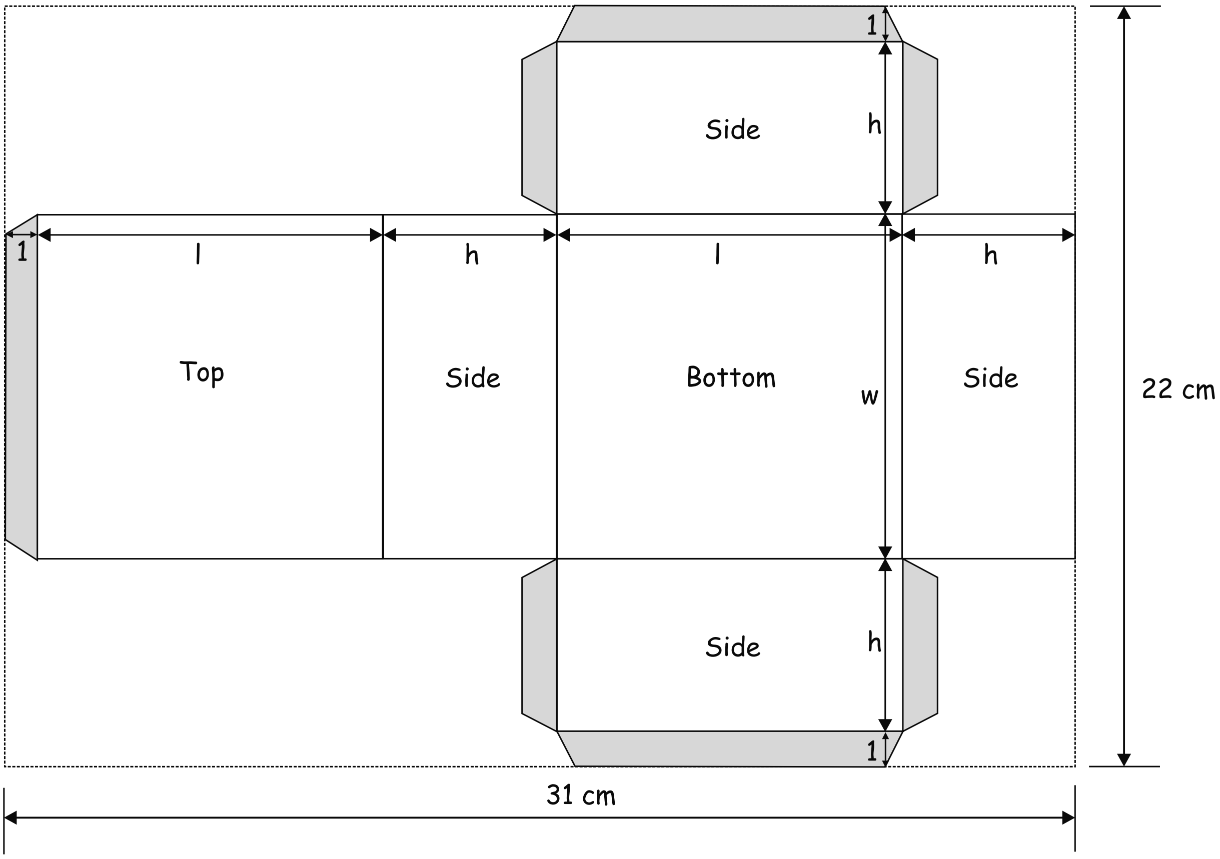
$h'(x) = x \times \frac{1}{x} + (\ln x) \times 1 = 1 + \ln x$

(ii) $h'(x) = 1 + \ln x \Rightarrow \ln x = h'(x) - 1$

$\int \ln x\,dx = \int (h'(x) - 1)\,dx = h(x) - x + c = x\ln x - x + c$

Question 7 (50 marks)

Question 7 (a)



$$2h + 2l + 1 = 31$$
$$2l + 2h = 30$$
$$l + h = 15$$
$$l = (15 - h) \text{ cm}$$

$$2h + w + 2 = 22$$
$$2h + w = 20$$
$$w = (20 - 2h) \text{ cm}$$

Question 7 (b)

$$V = l \times b \times h = (15 - h)(20 - 2h)h$$

Question 7 (c)

Square bottom:

$$l = w$$
$$15 - h = 20 - 2h$$
$$2h - h = 20 - 15$$
$$\therefore h = 5 \text{ cm}$$

$$V = (15 - (5))(20 - 2(5))(5)$$
$$= (10)(10)(5)$$
$$= 500 \text{ cm}^3$$

Question 7 (d)

$(15-h)(20-2h)h=500$ [Form a cubic equation by putting the volume of the box equal to 500.]

$(300-50h+2h^2)h=500$

$2h^3-50h^2+300h-500=0$

$h^3-25h^2+150h-250=0$

$h = 5$ is a solution of this cubic. Therefore, $(h-5)$ is a linear factor. The other factor is a quadratic. Find the quadratic by lining up.

$h^3-25h^2+150h-250=(h-5)(h^2+kh+50)$

$h^3-25h^2+150h-250=h^3+(k-5)h^2+(50-5k)h-250$

$\therefore -25=k-5 \Rightarrow k=-20$

$h^3-25h^2+150h-250=(h-5)(h^2-20h+50)=0$

$h^2-20h+50=0$ [Solve the quadratic using the formula.] $\boxed{x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}}$

$a=1,\ b=-20,\ c=50$

$$h=\frac{-(-20)\pm\sqrt{(-20)^2-4(1)(50)}}{2(1)}$$

$$=\frac{20\pm\sqrt{400-200}}{2}$$

$$=\frac{20\pm\sqrt{200}}{2}$$

$$=\frac{20\pm10\sqrt{2}}{2}$$

$$=10\pm5\sqrt{2}$$

$=\cancel{17.1},\ 2.9\,\text{cm}$ [Discard the solution of $h = 17.1$ cm. The length l of the box is equal to $(15-h)$. The length is not long enough to accomodate a value of 17.1 cm.]

Answer: $h = 2.9$ cm

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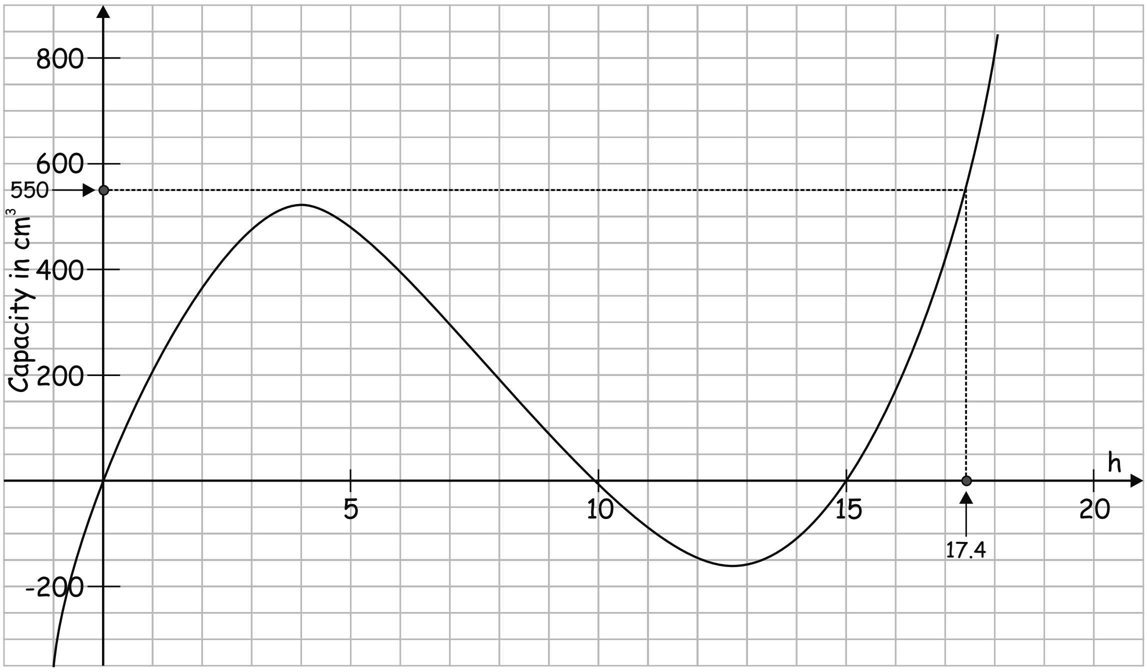
Question 7 (e)



Explanation:

10% extra volume: $500 \times 1.1 = 550 \text{ cm}^3$
Go to 550 cm^3 on the y-axis and read off the h value.
$\therefore h \approx 17.4$ cm

A height h = 17.4 cm is too long to make a box from this piece of cardboard. The length l of the box is equal to $(15 - h)$. The length is not long enough to accomodate a value of 17.4 cm.

Question 8 (50 marks)

Question 8 (a)

$P = ?$

$F = €20\ 000$

$t = 1$ year

$i = 0.03$

$$P = \frac{F}{(1+i)^t} = \frac{20\ 000}{(1+0.03)^1} = €19\ 417.48 \quad \boxed{F = P(1+i)^t}$$

Question 8 (b)

$$P = \frac{20\ 000}{(1.03)^t}$$

Question 8 (c)

I need to calculate the retirement fund that he has saved for and is available on the day of his retirement. €20 000 is drawn down immediately. The next €20 000 will not be drawn down for another year so its present value on the day of retirement is $\frac{20\ 000}{(1.03)^1}$. The next €20 000 will not be drawn from the retirement fund for 2 years so its present value is $\frac{20\ 000}{(1.03)^2}$. An so on.

$$20\ 000+\frac{20\ 000}{(1.03)^1}+\frac{20\ 000}{(1.03)^2}+............+\frac{20\ 000}{(1.03)^{24}}$$

$$=20\ 000\left[1+\frac{1}{1.03}+\frac{1}{1.03^2}+....................+\frac{1}{1.03^{24}}\right] \quad \boxed{S_n=\frac{a(1-r^n)}{1-r}}$$

$$a=1,\ r=\frac{1}{1.03},\ n=25$$

$$S_n=20\ 000\left[\frac{1(1-(\frac{1}{1.03})^{25})}{1-\frac{1}{1.03}}\right]=€358\ 710.84$$

Question 8 (d)

(i) $(1+i)^{12}=1.03$

$1+i=1.03^{\frac{1}{12}}=1.002466$

$\therefore i=0.002466=0.2466\%$

(ii) $F=P(1.002466)^n$

His first payment P will be compounded 480 times at an interest rate of 0.2466%. His second payment P will be compounded 479 times at an interest rate of 0.2466%. And so on.

(iii) $358\ 710.84=P(1.002466)^{480}+...........+P(1.002466)^1$

$$=\frac{P(1.002466)(1-(1.002466)^{480})}{1-1.002466}$$

$\therefore P=€390.17$

Question 8 (e)

10 years less: $30\times12=360$ months

$$358\ 710.84=\frac{P(1.002466)(1-(1.002466)^{360})}{1-1.002466}$$

$\therefore P=€618.35$

Set D Paper 1

QUESTION 9 (50 MARKS)

Question 9 (a)

(i) $f(x) = -0.5x^2 + 5x - 0.98$

$f(0.2) = -0.5(0.2)^2 + 5(0.2) - 0.98 = 0$

(ii) $f(x) = -0.5x^2 + 5x - 0.98$

$f'(x) = -x + 5 = 0 \Rightarrow x = 5$

$f(5) = -0.5(5)^2 + 5(5) - 0.98 = 11.52 \Rightarrow (5, 11.52)$ is a turning point.

$f''(x) = -1 < 0 \Rightarrow (5, 11.52)$ is a local maximum.

Question 9 (b) (i)

$f(x) = -0.5x^2 + 5x - 0.98$

$f(1) = -0.5(1)^2 + 5(1) - 0.98 = 3.52$

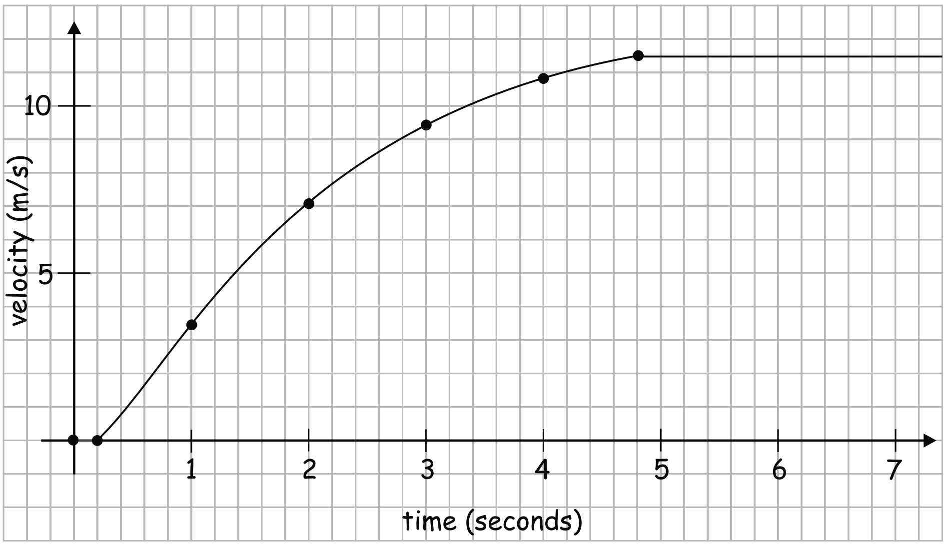
$f(2) = -0.5(2)^2 + 5(2) - 0.98 = 7.02$

[Work out a number of points for the middle part of the graph and then plot it.]

$f(3) = -0.5(3)^2 + 5(3) - 0.98 = 9.52$

$f(4) = -0.5(4)^2 + 5(4) - 0.98 = 11.02$

$f(5) = -0.5(5)^2 + 5(5) - 0.98 = 11.52$



Question 9 (b) (ii)

In the first 0.2 s the sprinter does not move. He/she is still in the blocks. To find the distance travelled over the first 5 s find the area under the velocity curve from 0.2 s to 5 s.

$$\boxed{v = \frac{ds}{dt} \Rightarrow s = \int v\,dt}$$

$$s = \int v\,dt$$

$$= \int (-0.5t^2 + 5t - 0.98)\,dt$$

$$= \frac{-0.5t^3}{3} + \frac{5t^2}{2} - 0.98t + c$$

To find the distance s travelled replace t by 5 and then t by 0.2 and subtracts the answers.

$$s = \left[\frac{-0.5(5)^3}{3} + \frac{5(5)^2}{2} - 0.98(5) + c\right] - \left[\frac{-0.5(0.2)^3}{3} + \frac{5(0.2)^2}{2} - 0.98(0.2) + c\right] = 36.9 \text{ m}$$

Question 9 (b) (iii)

The sprinter runs 36.9 m over the first 5 seconds.
He/she runs the rest of the race (63.1 m) at the maximum speed of 11.52 m/s.

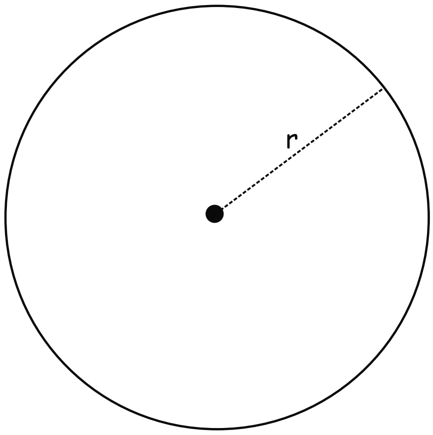
$$11.52 = \frac{63.1}{t} \Rightarrow t = \frac{63.1}{11.52} = 5.48 \text{ s} \qquad \boxed{v = \frac{s}{t}}$$

Answer: Total time = 10.48 s

Question 9 (c)

$\boxed{V = \frac{4}{3}\pi r^3}$

$\boxed{\text{Surface Area} = 4\pi r^2}$



(i) $$\frac{dV}{dt} \propto -A \Rightarrow \frac{dV}{dt} = -kA$$

$$\frac{d(\frac{4}{3}\pi r^3)}{dt} = -k(4\pi r^2)$$

$$\tfrac{4}{3}\pi r^2 \frac{dr}{dt} = -k(4\pi r^2)$$

$$\frac{dr}{dt} = -k$$

(ii) $$\frac{dr}{dt} = -k \Rightarrow \int dr = -k\int dt$$

$$r = -kt + c$$

$$t = 0 : r = r_0 \Rightarrow c = r_0$$

$$\therefore r = -kt + r_0$$

$$t = 1 : \tfrac{2}{3}\pi r_0^2 = \tfrac{4}{3}\pi r^3 \Rightarrow r = \frac{r_0}{2^{\frac{1}{3}}}$$

$$\therefore \frac{r_0}{2^{\frac{1}{3}}} = -k(1) + r_0 \Rightarrow k = \left(1 - \frac{1}{2^{\frac{1}{3}}}\right) r_0$$

$$t = T : r = 0 \Rightarrow 0 = -kT + r_0$$

$$\therefore T = \frac{r_0}{k} = \frac{r_0}{\left(1 - \frac{1}{2^{\frac{1}{3}}}\right) r_0} = 4.847 \text{ hours} \approx 291 \text{ minutes}$$

Set D Paper 1

SEC Set D (SP 2014): Paper 2

Question 1 (25 marks)

Question 1 (a)

1 – 0.383 – 0.575 – 0.004 = 0.038

x	13	14	15	16
$P(X = x)$	0·383	0·575	**0·038**	0·004

$$E(X) = \sum xP(x) = 13 \times 0.383 + 14 \times 0.575 + 15 \times 0.038 + 16 \times 0.004 = 13.663$$

Question 1 (b)

$E(X)$ represents the mean value of the age of all the second year students on 1 January 2010.

Question 1 (c)

$n = 10$
$r = 6$
$p(14) = 0.575$
$q(\text{Not } 14) = 0.425$

Bernoulli Trials
$p = P(\text{Success})$, $q = P(\text{Failure})$
$P(r \text{ successes}) = {}^nC_r p^r q^{n-r}$

$$P(6 \text{ out of } 10 \text{ are } 14 \text{ years of age}) = {}^{10}C_6 \times (0.575)^6 (0.425)^4 = 0.248$$

Question 2 (25 marks)

Question 2 (a)

Stratified Sampling: A probability sampling technique where the entire population is divided into non-overlapping subgroups (strata) and the final subjects are randomly selected proportionally from the different strata.
An advantage of this method over simple random sampling is greater precision using samples of the same size.

Cluster Sampling: A probabality sampling technique in which the sampler takes several steps in choosing the sample population. Firstly, the population is divided into clusters. A simple random sample of clusters is then selected from all of these clusters.
Finally, individuals are selected randomly from each cluster. An advantage over a simple random sample is that it is much cheaper.

Question 2 (b) (i)

$$\text{Margin of error} = \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{1111}} = 0.03$$

Question 2 (b) (ii)

Null hypothesis H_o: $P = 0.23$

Alternative hypothesis H_1: $P \neq 0.23$

Sample proportion $P = \dfrac{234}{1111} = 0.2106$

Confidence interval $= [0.2106 - 0.03, 0.2106 + 0.03] = [0.1806,\ 0.2406]$

There is evidence to support the party's claim that it has the support of 23% of the electorate because, based on the sample data, any values in the range 18% – 24% are possible values for the proportion of the electorate who support the party.
23% is in this confidence interval. Therefore, you cannot reject the null hypothesis.

Question 3 (25 marks)

Question 3 (a)

Is $P(4k-2,\ 3k+1) \in l_1 : 3x - 4y + 10 = 0$? [To see if a point is on a line, substitute the point into the equation of the line and show it satisfies the equation.]

$3(4k-2) - 4(3k+1) + 10$

$= 12k - 6 - 12k - 4 + 10$

$= 0$

Question 3 (b)

$l_1 : 3x - 4y + 10 = 0 \Rightarrow m_1 = \frac{3}{4}$ $\boxed{m_1 \times m_2 = -1}$ [The product of the slopes of perpendicular lines is −1.]

$l_2 : m_2 = -\frac{4}{3}$

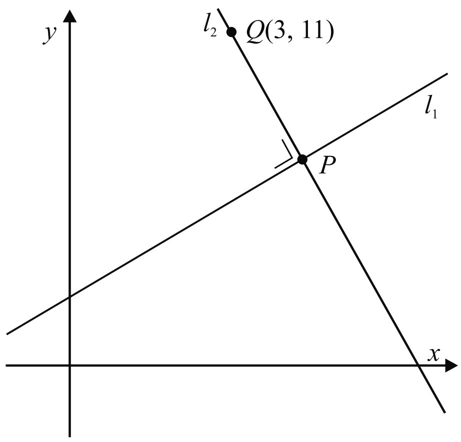
Equation of l_2: Point $P(4k - 2, 3k + 1)$, $m_2 = -\frac{4}{3}$ $\boxed{y - y_1 = m(x - x_1)}$

$y - (3k+1) = -\frac{4}{3}(x - (4k-2))$

$3(y - 3k - 1) = -4(x - 4k + 2)$

$3y - 9k - 3 = -4x + 16k - 8$

$4x + 3y - 25k + 5 = 0$



Question 3 (c)

$Q(3, 11) \in l_2 : 4x + 3y - 25k + 5 = 0$

$\therefore 4(3) + 3(11) - 25k + 5 = 0$

$12 + 33 - 25k + 5 = 0$

$50 = 25k$

$\therefore k = 2$

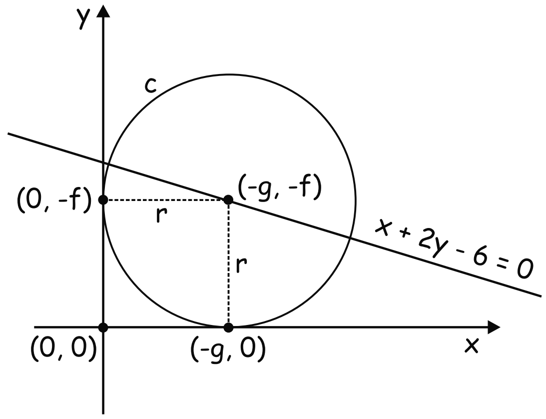
Question 3 (d)

P is the point at the foot of the perpendicular.

$P(4k-2,\ 3k+1) = P(4(2)-2,\ 3(2)+1) = P(6,\ 7)$

Set D Paper 2

QUESTION 4 (25 MARKS)



CENTRE AND RADIUS OF A CIRCLE
$c: x^2 + y^2 + 2gx + 2fy + c = 0$
Centre: $(-g, -f)$
$r = \sqrt{g^2 + f^2 - c}$

Sketch the situation.
The circle c touches the x-axis at $(-g, 0)$ and touches the y-axis at $(0, -f)$.
The line passes through the centre of the circle c.

$(-g, -f) \in x + 2y - 6 = 0$ [If a point is on a line, substitute the point into the equation of the line and it satisfies the equation.]

$\therefore (-g) + 2(-f) - 6 = 0$

$-g - 2f - 6 = 0$

$g + 2f = -6 ....(\mathbf{1})$

The points $(-g, 0)$ and $(0, -f)$ satisfy the equation of the circle.

$(-g, 0) \in c: x^2 + y^2 + 2gx + 2fy + c = 0$

$\therefore (-g)^2 + (0)^2 + 2g(-g) + 2f(0) + c = 0$

$g^2 - 2g^2 + c = 0$

$c = g^2$

$(0, -f) \in c: x^2 + y^2 + 2gx + 2fy + c = 0$

$\therefore (0)^2 + (-f)^2 + 2g(0) + 2f(-f) + c = 0$

$f^2 - 2f^2 + c = 0$

$c = f^2$

$$\therefore c = g^2 = f^2 .....(\mathbf{2})$$

$$\therefore g^2 = f^2 \Rightarrow g = \pm f .....(\mathbf{3})$$

To find the equation of each circle, choose $g = +f$ for the first circle equation and $g = -f$ for the second circle equation.

$g = +f$ [Substitute into Eqn. (**1**)]

$g + 2g = -6$

$3g = -6$

$g = -2$

$f = -2$ [Substitute into Eqn. (**2**)]

$c = g^2 = (-2)^2 = 4$

$g = -f$

$-f + 2f = -6$

$f = -6$

$g = 6$

$c = g^2 = (6)^2 = 36$

EQUATION OF CIRCLES

$x^2 + y^2 + 2(-2)x + 2(-2)y + 4 = 0 \Rightarrow x^2 + y^2 - 4x - 4y + 4 = 0$

$x^2 + y^2 + 2(6)x + 2(-6)y + 36 = 0 \Rightarrow x^2 + y^2 + 12x - 12y + 36 = 0$

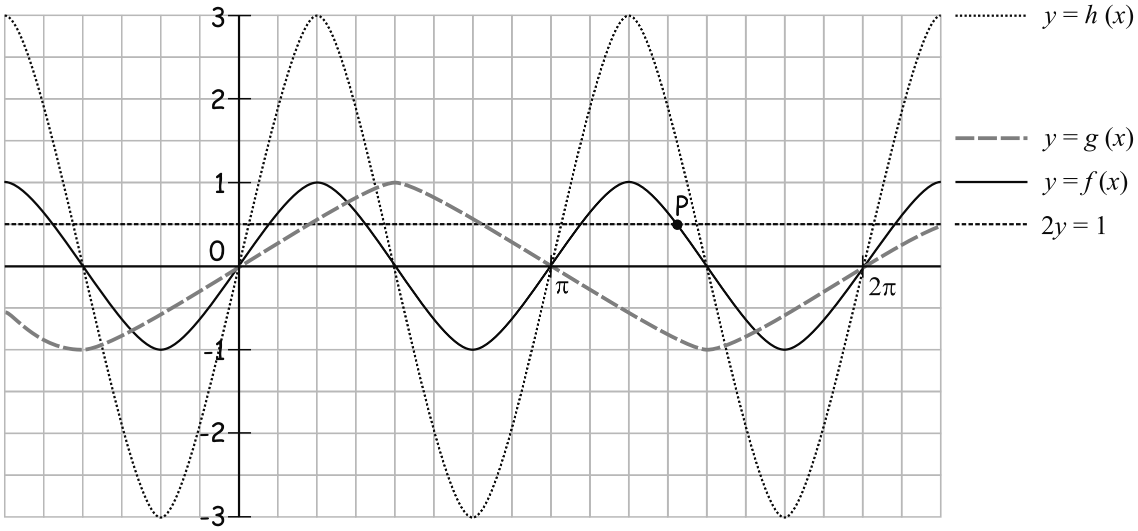
Question 5 (25 marks)

Question 5 (a)

$g: x \mapsto \sin x \Rightarrow P = 2\pi,\ R = [-1, 1]$

$h: x \mapsto 3\sin 2x \Rightarrow P = \dfrac{2\pi}{2} = \pi,\ R = [-3, 3]$

$$y = a\sin nx$$
$$R = [-a, a],\ P = \frac{2\pi}{n}$$



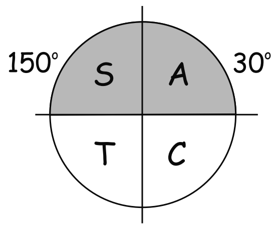
Question 5 (b)

$y = f(x) = \sin 2x$

$2y = 1 \Rightarrow y = \frac{1}{2}$

$\therefore \sin 2x = \frac{1}{2}$

$2x = \sin^{-1}(\frac{1}{2}) = 30^\circ = \dfrac{\pi}{6}$



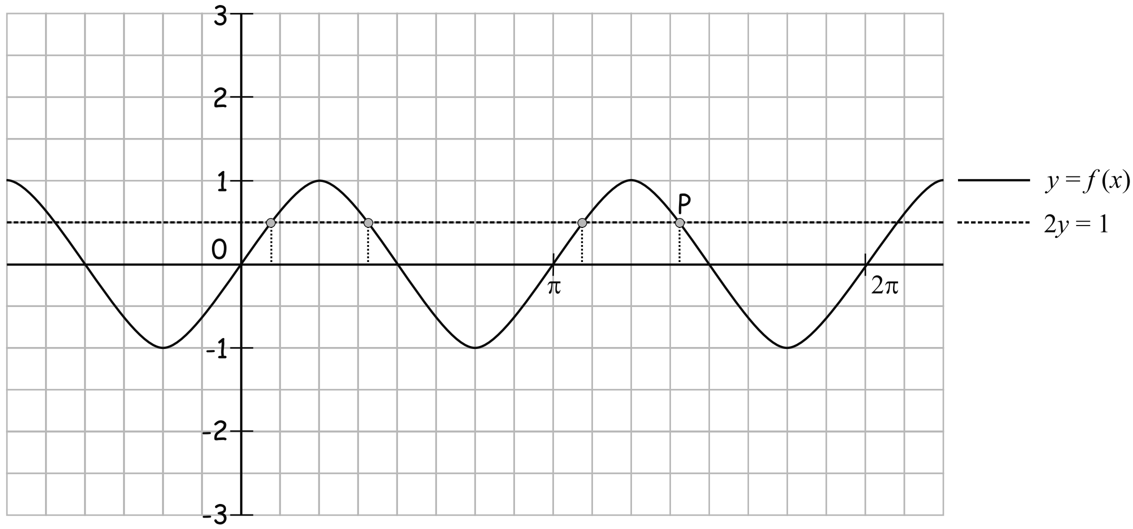
$2x = \dfrac{\pi}{6}, \dfrac{13\pi}{6}$ [First quadrant]

$2x = \dfrac{5\pi}{6}, \dfrac{17\pi}{6}$ [Second quadrant]

$x = \dfrac{\pi}{12}, \dfrac{13\pi}{12}$

$x = \dfrac{5\pi}{12}, \dfrac{17\pi}{12}$

$\therefore x = \dfrac{\pi}{12}, \dfrac{5\pi}{12}, \dfrac{13\pi}{12}, \dfrac{17\pi}{12}$ [These are all the solutions between 0 and 2π.]



Answer: $P\left(\dfrac{17\pi}{12}, \dfrac{1}{2}\right)$

QUESTION 6A (25 MARKS)

Explanation:

Proof by contradiction is a form of proof that establishes the truth or validity of a proposition by showing that the proposition being false would imply a contradiction.

Example:

Prove $x+\frac{1}{x}\geq 2$ for all $x>0,\ x\in\mathbb{R}$.

To prove this let's assume it is false, i.e $x+\frac{1}{x}<2$ for all $x>0,\ x\in\mathbb{R}$.

$x+\frac{1}{x}<2$

$x^2+1<2x$

$x^2-2x+1<0$

$(x-1)^2<0$ [This statement is false for all values of x.]

This is a contradiction.

QUESTION 6B (25 MARKS)

OECD is a cyclic quadrilateral because its opposite angles add up to 180°.

$|\angle OEC|+|\angle ODC|=90^\circ+90^\circ=180^\circ$

It follows that the other pair of opposite angles also add up to 180° as the four angles in a quadrilateral add up to 360°.

CYCLIC QUADRILATERALS

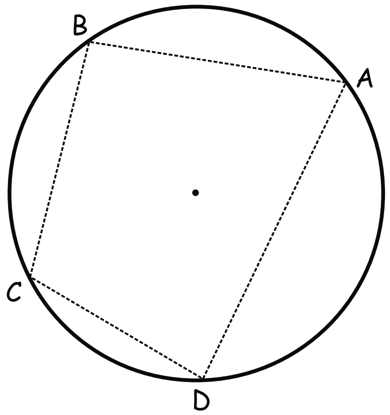
A cyclic quadilateral is a four sided figure whose vertices lie on a circle.

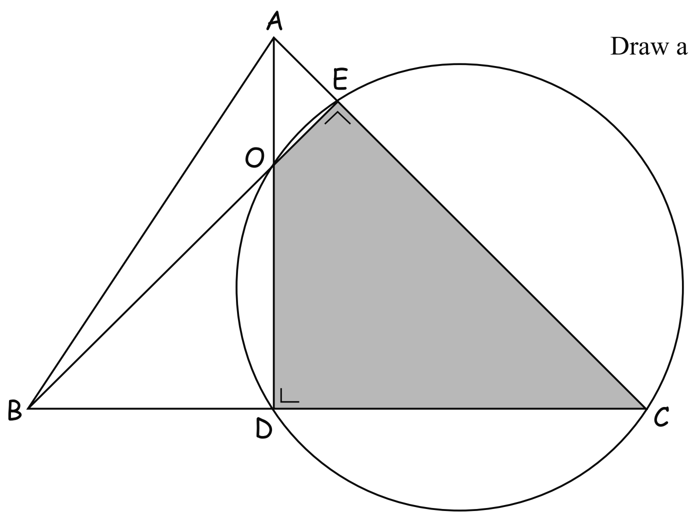
Opposite angles of a cyclic quadrilateral add up to 180°.

$$\angle A+\angle C=180^\circ$$

$$\angle B+\angle D=180^\circ$$

Conversely, if the opposite angles of a quadrilateral add up to 180° then it is a cyclic quadrilateral.





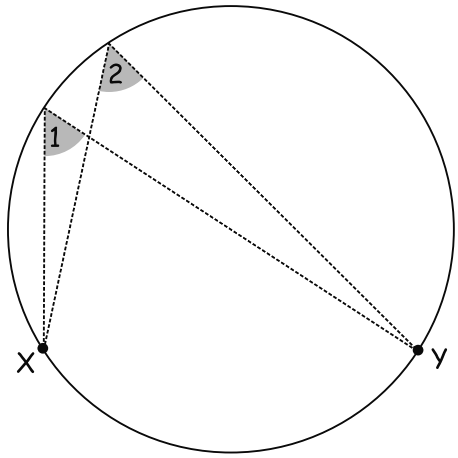
Draw a circle around the quadrilateral.

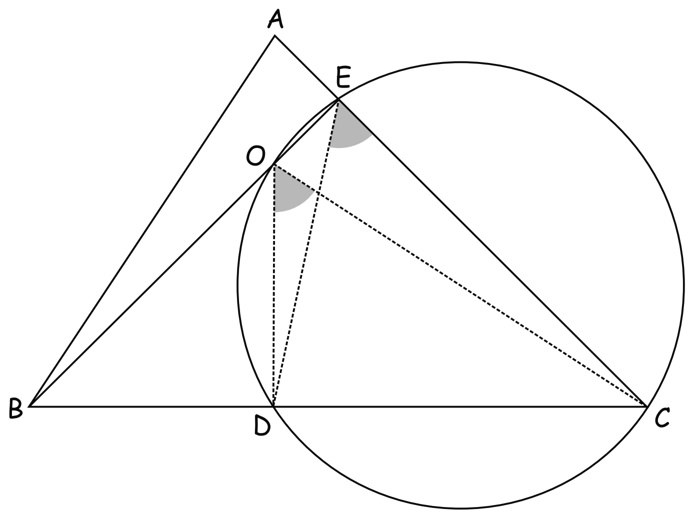
ANGLES STANDING ON THE SAME ARC IN A CIRCLE

Angles 1 and 2 are standing on the same arc [AB].

Angles standing on the same arc are equal.

$\therefore |\angle 1| = |\angle 2|$.





$|\angle DOC| = |\angle DEC|$ [Both angles are standing on arc [DC]]

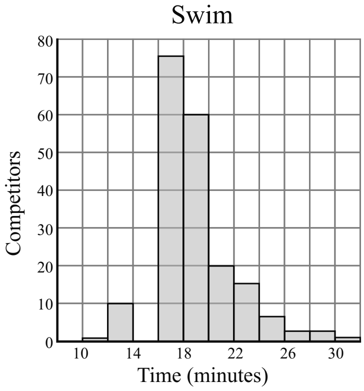
Set D
Paper 2

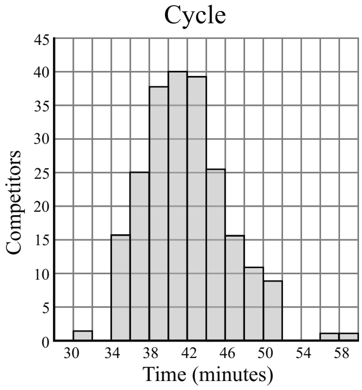
Question 7 (75 marks)

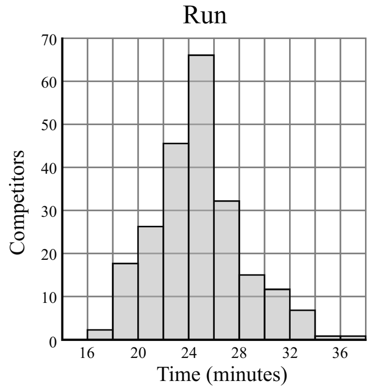
Question 7 (a)

(i)

	Swim	*Cycle*	*Run*
Mean	18.329	41.927	**?**
Median	17.900	41.306	**?**
Mode	#N/A	#N/A	#N/A
Standard Deviation	**?**	4.553	3.409
Sample Variance	10.017	20.729	11.622
Skewness	1.094	0.717	0.463
Range	19.226	27.282	20.870
Minimum	**11.350**	**31.566**	**16.466**
Maximum	**30.576**	**58.847**	**37.336**
Count	224	224	224



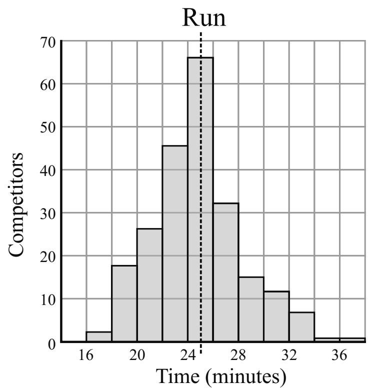




You can match the histograms to the events by looking at the maximum and minimum times to complete each event.

(ii) The median is a line on the histogram that bisects its area. The area to the left of it is equal to the area to the right of it. Using your eye this appears to lie along the class interval of 24–26 minutes.

mean ≈ median ≈ 25 minutes



(iii) Swim: Sample Variance s = 10.017

$$\boxed{s = \sigma^2}$$

$$\sigma = \sqrt{s} = \sqrt{10.017} = 3.16 \text{ minutes}$$

(iv) There was probably no discrete modal result as all times were different or there were lots of the same times. Therefore, there is no modal time. There is a modal class alright but no modal time.

QUESTION 7 (b)

Cycle vs. Swim: Moderate positive correlation
Run vs. Swim: Moderately strong positive correlation
Run vs. Cycle: Strong positive correlation

QUESTION 7 (c)

Run/Swim: $y = 0.53x + 15.2$
Brian: $x = 17.6$ mins
$y = 0.53(17.6) + 15.2 = 24.528$ mins

Run/Cycle: $y = 0.58x + 0.71$
Brian: $x = 35.7$ mins
$y = 0.58(35.7) + 0.71 = 21.416$ mins

Take the average of the 2 run times: $\frac{24.528+21.416}{2} = 22.972$ mins

The mean finishing time for the overall event was 88·1 minutes and the standard deviation was 10·3 minutes.

QUESTION 7 (d)

In any normal distribution with mean $\bar{x}$ and standard deviation σ.
1. 68.26% of the data falls within 1σ of the mean $\bar{x}$.
2. 95.46% of the data falls within 2σ of the mean $\bar{x}$.
3. 99.74% of the data falls within 3σ of the mean $\bar{x}$.

$\bar{x} = 88.1$ mins, $\sigma = 10.3$ mins

$\bar{x} - 2\sigma = 88.1 - 2(10.3) = 67.5$ mins

$\bar{x} + 2\sigma = 88.1 + 2(10.3) = 108.7$ mins

"95% of the athletes took between **67.5** and **108.7** minutes to complete the race."

QUESTION 7 (e)

$P(x < 100) = P(z < 1.155) = 0.877$

$\boxed{z = \frac{x - \bar{x}}{\sigma}}$ $\quad z = \frac{100 - 88.1}{10.3} = 1.155$

Number of athletes $= 224 \times 0.877 \approx 196$

QUESTION 7 (f)

Let p = Probabilty of completing the race in less than 100 minutes = $p(0.877)$
Let q = Probabilty of completing the race in more than 100 minutes = $q(1 – 0.877) = q(0.123)$

This is the order in which she interviews the athletes:

Fixed

$q \quad p \quad p \quad p \quad \boxed{p} \quad q$

No. of permutations $= \frac{5!}{4!} = 5$

BERNOULLI TRIALS
$p = P(\text{Success})$, $q = P(\text{Failure})$
$P(r \text{ successes}) = {}^{n}C_{r}\, p^{r} q^{n-r}$

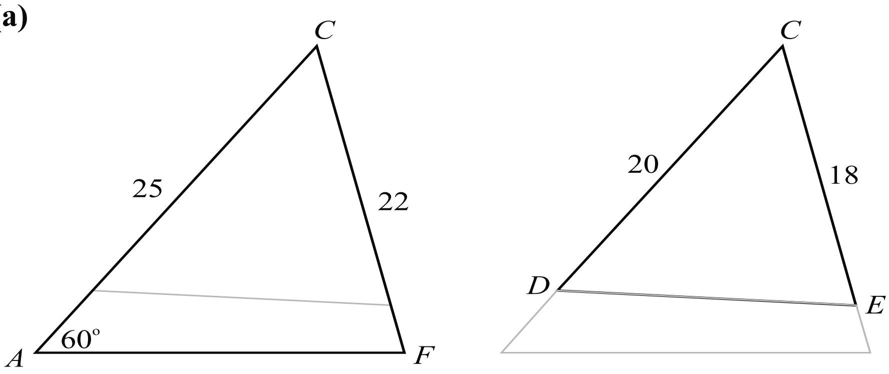
$P = (0.123)^2 \times (0.877)^4 \times 5 = 0.0447$

The probability that the second person she interviews will be the sixth person she approaches is about 4.5%.

Set D Paper 2

Question 8 (50 marks)

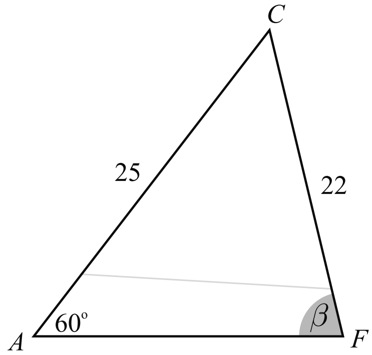
Question 8 (a)



Call β the measure of $|\angle CFA|$. Use the Sine Rule to find this angle.

$$\boxed{\frac{\sin A}{a} = \frac{\sin B}{b}}$$

[Use the Sine Rule anytime you are given 2 sides and a non-included angle.]



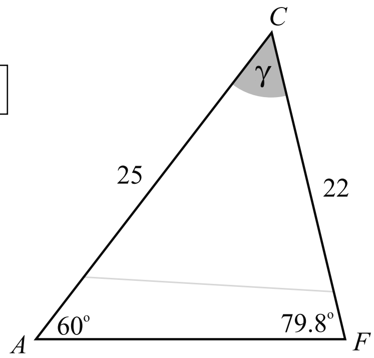
$$\frac{\sin \beta}{25} = \frac{\sin 60^\circ}{22}$$

$$\therefore \sin \beta = \frac{25 \sin 60^\circ}{22}$$

$$\beta = \sin^{-1}\left(\frac{25 \sin 60^\circ}{22}\right) = 79.8^\circ$$

Call γ, the measure of $|\angle ACF|$.

$$\boxed{\text{The 3 angles of a triangle add up to } 180^\circ.}$$



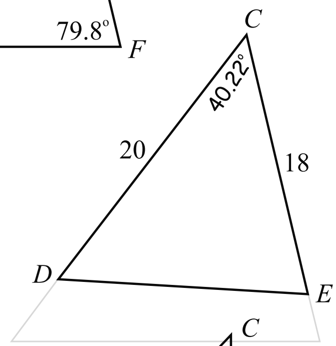
$$\gamma + 60^\circ + 79.78^\circ = 180^\circ$$

$$\gamma = 180^\circ - 60^\circ - 79.8^\circ = 40.2^\circ$$

Use the Cosine Rule to find $|DE|$.

$$\boxed{a^2 = b^2 + c^2 - 2bc \cos A}$$

[Use the Cosine Rule anytime you are given 2 sides and an included angle.]



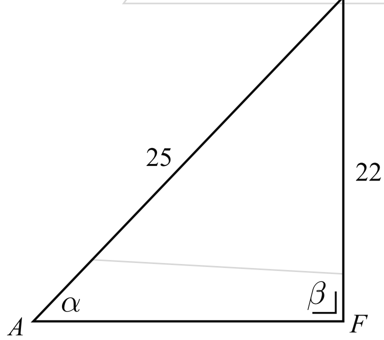
$$|DE|^2 = 20^2 + 18^2 - 2(20)(18)\cos 40.2^\circ$$

$$|DE| = \sqrt{20^2 + 18^2 - 2(20)(18)\cos 40.2^\circ} = 13.2 \text{ cm}$$

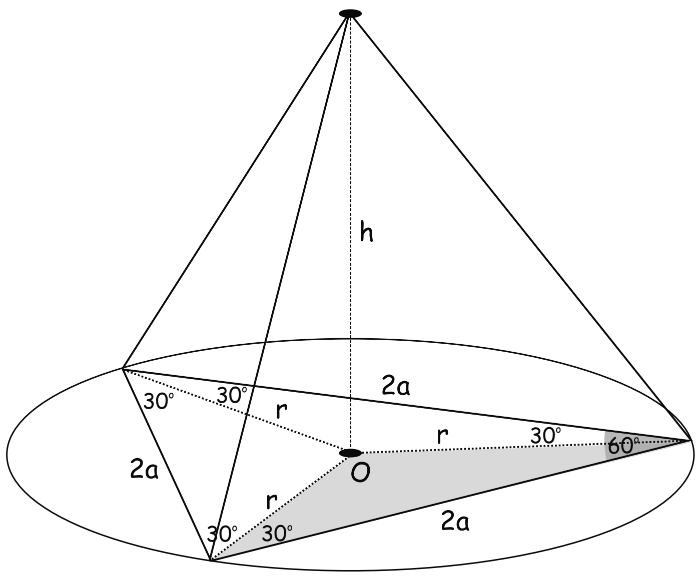
Question 8 (b)

$\sin \alpha = \frac{22}{25} \Rightarrow \alpha = \sin^{-1}(\frac{22}{25}) = 62^\circ$

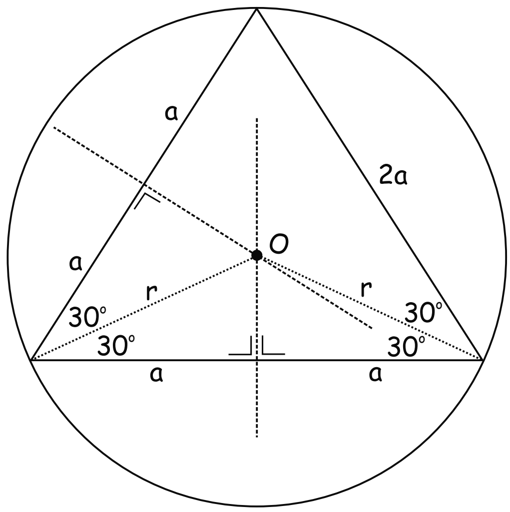
The maximum of the angle β is 90°. If it goes beyond 90° the solar panel will topple over.



QUESTION 9 (25 MARKS)



An equilateral triangle fits into the base of the cylinder exactly. The circle at the base of the cylinder is a circumcircle with centre O. The centre of a circumcircle O is found by intersecting the perpendicular bisectors of the sides of the triangle. Lift out the highlighted triangle to find an expression for r.

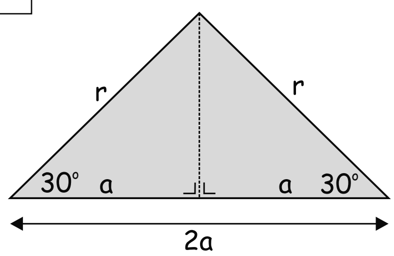


$$\cos A = \frac{\text{Adjacent}}{\text{Hypotenuse}}$$

$$\cos 30^\circ = \frac{a}{r}$$

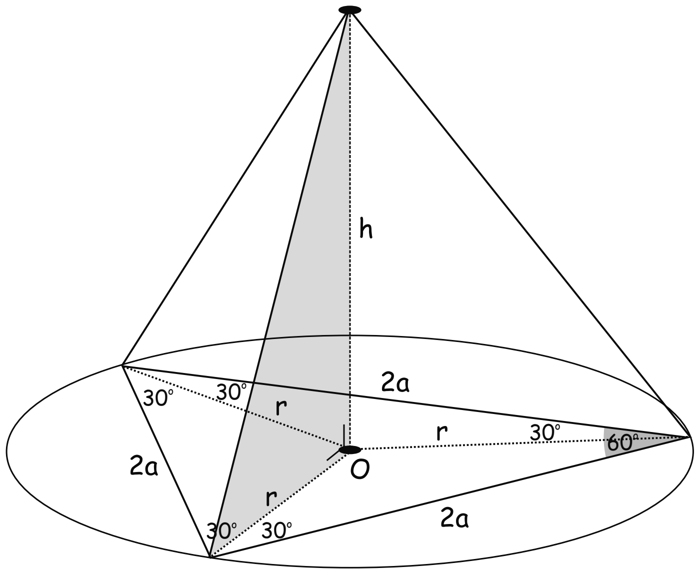
$$\frac{\sqrt{3}}{2} = \frac{a}{r}$$

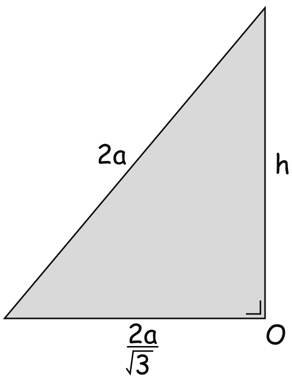
$$\therefore r = \frac{2a}{\sqrt{3}}$$



Set D Paper 2

Lift out the highlighted triangle to find an expression for *h*. It is a right-angled triangle.





$\boxed{a^2+b^2=c^2}$ Use Pythagoras.

$$h^2+\left(\frac{2a}{\sqrt{3}}\right)^2=(2a)^2$$

$$h^2+\frac{4a^2}{3}=4a^2$$

$$h^2=4a^2-\frac{4a^2}{3}=\frac{8a^2}{3}$$

$$h=\sqrt{\frac{8a^2}{3}}=\frac{2\sqrt{2}a}{\sqrt{3}}$$

$\boxed{V=\pi r^2 h}$ Volume of a Cylinder

$$\begin{aligned}V&=\pi r^2 h\\&=\pi\left(\frac{2a}{\sqrt{3}}\right)^2\left(\frac{2\sqrt{2}a}{\sqrt{3}}\right)\\&=\pi\left(\frac{4a^2}{3}\right)\left(\frac{2\sqrt{2}a}{\sqrt{3}}\right)\\&=\frac{8\pi\sqrt{2}a^3}{3\sqrt{3}}\\&=\frac{8\pi\sqrt{2}a^3}{3\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}}\\&=\left(\frac{8\sqrt{6}}{9}\right)\pi a^3\end{aligned}$$